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FORMAL SYMPLECTIC GROUPOID OF A DEFORMATION QUANTIZATION

ALEXANDER V. KARABEGOV

ABSTRACT. We give a self-contained algebraic description of a formal symplectic groupoid over a Poisson manifold M. To each natural star product on M we then associate a canonical formal symplectic groupoid over M. Finally, we construct a unique formal symplectic groupoid 'with separation of variables' over an arbitrary Kähler-Poisson manifold.

1. INTRODUCTION

Symplectic groupoids are semiclassical geometric objects whose heuristic quantum counterparts are associative algebras treated as quantum objects. In [4], [14], and [15] evidence was given that the star algebra of a deformation quantization gives rise to a formal analogue of a symplectic groupoid. In this paper we give a global definition of a formal symplectic groupoid and show that to each natural deformation quantization (in the sense of Gutt and Rawnsley, [10]) there corresponds a canonical formal symplectic groupoid.

Symplectic groupoids were introduced independently by Karasëv [16], Weinstein [23], and Zakrzewski [25]. Recall that a local symplectic groupoid is an object that has the properties of a symplectic groupoid in which the multiplication is local, being only defined in a neighborhood of the unit space. It was proved in [16] and [23] that for any Poisson manifold M there exists a local symplectic groupoid over M that 'integrates' it. In [4] A. S. Cattaneo, B. Dherin, and G. Felder considered the formal integration problem for $\mathbb{R}^n$ endowed with an arbitrary Poisson structure, whose solution is given by a formal symplectic groupoid. They start with the zero Poisson structure on $\mathbb{R}^n$, the corresponding trivial symplectic groupoid $T^*\mathbb{R}^n$, and a generating function of the Lagrangian product space of this groupoid. A formal symplectic groupoid is then defined in terms of a formal deformation

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of that trivial generating function. One of the main results of [4] is an explicit formula for a generating function that delivers the formal symplectic groupoid related to the Kontsevich star product. The approach to formal symplectic groupoids developed in [4] demonstrates the relationship between geometric and algebraic deformations described in [22].

One can take an alternative approach to the definition of a formal symplectic groupoid over a Poisson manifold M (which leads to the same object) by replacing the symplectic manifold Σ on which a (local) symplectic groupoid over M is defined, with the formal neighborhood (Σ, Λ) of its unit space Λ (see the definition of a formal neighborhood in Section 2). We use a simple model of the algebra of formal functions on (Σ, Λ) which is reminiscent of the Hopf algebroid constructed by Vainerman in [21]. This model provides effective means to check the axioms of a formal symplectic groupoid and to do the calculations.

In Section 2 we state formal analogues of the axioms of a symplectic groupoid and give a definition of a formal symplectic groupoid over a given Poisson manifold M. Such a formal groupoid is defined on the formal neighborhood of a Lagrangian submanifold of a symplectic manifold. In Section 3 we give a self-contained algebraic description of a formal symplectic groupoid and show that a strict formal symplectic realization of an arbitrary Poisson manifold M gives rise to a unique formal symplectic groupoid over M whose source mapping is given by that formal symplectic realization. In Section 4 we describe the space of all formal symplectic groupoids over M which are defined on a given formal neighborhood of a Lagrangian submanifold of a symplectic manifold. In Section 5 we relate to each natural deformation quantization on M a canonical formal symplectic groupoid. In Section 6 we prove that any deformation quantization with separation of variables on a Kähler-Poisson manifold M is natural and show that its canonical formal symplectic groupoid has a property which we call 'separation of variables'. Finally, in Section 7 we prove that for an arbitrary Kähler-Poisson manifold M there exists a unique formal symplectic groupoid with separation of variables over M.

2. Definition of a formal symplectic groupoid

A symplectic groupoid over a Poisson manifold $(M, \{\cdot,\cdot\}_M)$ is a symplectic manifold Σ endowed with the associated Poisson source mapping $s : \Sigma \to M$, the anti-Poisson target mapping $t : \Sigma \to M$, which both are surjective submersions, the antisymplectic involutive inverse mapping $i : \Sigma \to \Sigma$, and the unit mapping $\epsilon : M \to \Sigma$,

which is an embedding. The image $\Lambda = \epsilon(M)$ of the unit mapping is the Lagrangian unit space of the symplectic groupoid. Denote by Σ^n the Cartesian product of n copies of the manifold Σ and by Σ_n the submanifold of Σ^n formed by the n-tuples $(\alpha_1, \ldots, \alpha_n) \in \Sigma^n$ such that $t(\alpha_k) = s(\alpha_{k+1}),\ 1 \leq k \leq n-1$. The coisotropic submanifold Σ_2 of $\Sigma \times \Sigma$ is the domain of the groupoid multiplication $m : \Sigma_2 \to \Sigma$. For $\alpha, \beta \in \Sigma_2$ we write $m(\alpha, \beta) = \alpha\beta$. The groupoid multiplication is associative. For $(\alpha, \beta, \gamma) \in \Sigma_3$ the associativity condition $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ holds. The graph $\Gamma = \{(\alpha, \beta, \gamma) \mid (\beta, \gamma) \in \Sigma_2, \alpha = \beta\gamma\}$ of the groupoid multiplication (the product space) is a Lagrangian submanifold of $\Sigma \times \bar{\Sigma} \times \bar{\Sigma}$, where $\bar{\Sigma}$ is a copy of the manifold Σ endowed with the opposite symplectic structure.

The groupoid operations satisfy the following axioms. For any composable $\alpha, \beta \in \Sigma$ and $x \in M$

$$(A1)\ s(\alpha\beta) = s(\alpha),\ (A2)\ t(\alpha\beta) = t(\beta),\ (A3)\ s \circ \epsilon = \mathrm{id}_M,$$
$$(A4)\ t \circ \epsilon = \mathrm{id}_M,\ (A5)\ \epsilon(s(\alpha))\alpha = \alpha, (A6)\ \alpha\epsilon(t(\alpha)) = \alpha,$$
$$(A7)\ s(i(\alpha)) = t(\alpha), (A8)\ \alpha i(\alpha) = \epsilon(s(\alpha)),\ (A9)\ i(\alpha)\alpha = \epsilon(t(\alpha)).$$

Recall the definition of the formal neighborhood (X, Y) of a submanifold Y of a manifold X. Let Y be a closed k-dimensional submanifold of a real n-dimensional manifold X and $I_Y \subset C^\infty(X)$ be the ideal of smooth functions on X vanishing on Y. Then the quotient algebra $C^\infty(X, Y) := C^\infty(X)/I_Y^\infty$, where $I_Y^\infty = \cap_{l=1}^\infty I_Y^l$, can be thought of as the algebra of smooth functions on the formal neighborhood (X, Y) of the submanifold Y in X. If $U \subset X$ is a local coordinate chart on X with coordinates $\{x^i\}$ such that $U \cap Y$ is given by the equations $x^{k+1} = 0, \ldots, x^n = 0$, then $C^\infty(U, U \cap Y)$ is isomorphic to $C^\infty(U \cap Y)[[x^{k+1}, \ldots, x^n]]$, where the isomorphism is established via the formal Taylor expansion of the functions on U in the variables $x^{k+1}, \ldots, x^n$. Thus the formal neighborhood (X, Y) of $Y \subset X$ is the ringed space Y with the sheaf of rings whose global sections form the algebra $C^\infty(X, Y)$. Let Y_i be a submanifold of a manifold X_i for $i = 1, 2$. If $f : X_1 \to X_2$ is a mapping such that $f(Y_1) \subset Y_2$, then $f^*(I_{Y_2}) \subset I_{Y_1}$. Therefore the mapping f induces the dual morphism of algebras $f^* : C^\infty(X_2, Y_2) \to C^\infty(X_1, Y_1)$.

Denote by Λ^n the Cartesian product of n copies of the manifold Λ and by Λ_n the diagonal of Λ^n. Notice that $\Sigma_n \cap \Lambda^n = \Lambda_n$. The algebra $C^\infty(\Sigma)$ is a Poisson algebra with respect to the natural Poisson bracket $\{\cdot, \cdot\}_\Sigma$ on Σ. The space $C^\infty(\Sigma, \Lambda)$ inherits a structure of Poisson algebra from $C^\infty(\Sigma)$. We will use the same notation $\{\cdot, \cdot\}_\Sigma$ for the induced Poisson bracket on $C^\infty(\Sigma, \Lambda)$. Similarly, denote by $\{\cdot, \cdot\}_{\Sigma^n}$ the Poisson

bracket on $C^\infty(\Sigma^n)$ corresponding to the product Poisson structure, and the induced bracket on $C^\infty(\Sigma^n,\Lambda^n)$. Let $\iota:\Sigma_2\to\Sigma\times\Sigma$ be the inclusion mapping. We will say that functions $F\in C^\infty(\Sigma)$ and $G\in C^\infty(\Sigma\times\Sigma)$ such that $m^*F=\iota^*G$ agree on Σ_2. The functions $F\in C^\infty(\Sigma)$ and $G\in C^\infty(\Sigma\times\Sigma)$ agree on Σ_2 if and only if the function $F\otimes 1\otimes 1-1\otimes G\in C^\infty(\Sigma\times\bar\Sigma\times\bar\Sigma)$ vanishes on the product space Γ. Since Γ is a Lagrangian submanifold of $\Sigma\times\bar\Sigma\times\bar\Sigma$, the Poisson bracket of two functions vanishing on Γ also vanishes on Γ. For functions $F_i\in C^\infty(\Sigma)$ and $G_i\in C^\infty(\Sigma\times\Sigma)$, $i=1,2$, the Poisson bracket of $F_1\otimes 1\otimes 1-1\otimes G_1$ and $F_2\otimes 1\otimes 1-1\otimes G_2$ equals

$$\{F_1,F_2\}_\Sigma\otimes 1\otimes 1-1\otimes\{G_1,G_2\}_{\Sigma^2},$$

whence we obtain the following lemma.

Lemma 1. *If functions $F_i\in C^\infty(\Sigma)$ and $G_i\in C^\infty(\Sigma\times\Sigma)$, $i=1,2$, agree on Σ_2, then the Poisson brackets $\{F_1,F_2\}_\Sigma$ and $\{G_1,G_2\}_{\Sigma^2}$ also agree on Σ_2.*

The multiplication $m:\Sigma_2\to\Sigma$ identifies Λ_2 with Λ and thus induces the comultiplication mapping

$$m^*:C^\infty(\Sigma,\Lambda)\to C^\infty(\Sigma_2,\Lambda_2).$$

Denote by $\iota_n:\Sigma_n\to\Sigma^n$ the inclusion mapping. In particular, $\iota=\iota_2$. Since the mapping ι_n maps Λ_n to Λ^n, it induces the algebra morphism

$$\iota_n^*:C^\infty(\Sigma^n,\Lambda^n)\to C^\infty(\Sigma_n,\Lambda_n).$$

We say that elements $F\in C^\infty(\Sigma,\Lambda)$ and $G\in C^\infty(\Sigma^2,\Lambda^2)$ agree on $C^\infty(\Sigma_2,\Lambda_2)$ if $m^*F=\iota^*G$ in $C^\infty(\Sigma_2,\Lambda_2)$. It follows from Lemma 1 that if $F_i\in C^\infty(\Sigma,\Lambda)$ agrees with $G_i\in C^\infty(\Sigma^2,\Lambda^2)$ on $C^\infty(\Sigma_2,\Lambda_2)$ for $i=1,2$, then $\{F_1,F_2\}_\Sigma$ agrees with $\{G_1,G_2\}_{\Sigma^2}$ on $C^\infty(\Sigma_2,\Lambda_2)$ as well. We will call this property of comultiplication *Property P* and use it in the definition of a formal symplectic groupoid. The mappings $s,t:\Sigma\to M$ induce the algebra morphisms

$$S,T:C^\infty(M)\to C^\infty(\Sigma,\Lambda).$$

The source mapping S is a Poisson morphism and the target mapping T is an anti-Poisson morphism. For any $f,g\in C^\infty(M)$ the elements $Sf,Tg\in C^\infty(\Sigma,\Lambda)$ Poisson commute. The unit mapping $\epsilon:M\to\Sigma$ identifies M with Λ and thus induces the algebra morphism $E:C^\infty(\Sigma,\Lambda)\to C^\infty(M)$. Axioms (A3) and (A4) imply that

$$ES=\mathrm{id}_{C^\infty(M)}\ \text{and}\ ET=\mathrm{id}_{C^\infty(M)}\,. \tag{1}$$

The inverse mapping $i:\Sigma\to\Sigma$ leaves fixed the elements of Λ and therefore induces the antisymplectic involutive algebra morphism I :

$C^\infty(\Sigma, \Lambda) \to C^\infty(\Sigma, \Lambda)$. It follows from Axiom (A7) that

$$IS = T. \tag{2}$$

To find the formal analogue of multiplication in a symplectic groupoid, we need a different description of the algebra $C^\infty(\Sigma_n, \Lambda_n)$. For $f \in C^\infty(M)$ introduce functions $S_n^k f, T_n^k f \in C^\infty(\Sigma^n, \Lambda^n)$ by the following formulas:

$$S_n^k f = \underbrace{1 \otimes \ldots \otimes \overbrace{Sf}^{k-\mathrm{th}} \otimes \ldots \otimes 1}_{n}, \; T_n^k f = \underbrace{1 \otimes \ldots \otimes \overbrace{Tf}^{k-\mathrm{th}} \otimes \ldots \otimes 1}_{n}. \tag{3}$$

Denote by $\mathcal{I}_n$ the ideal in $C^\infty(\Sigma^n, \Lambda^n)$ generated by the functions $S_n^{k+1} f - T_n^k f,\ f \in C^\infty(M),\ 1 \leq k \leq n-1$. Taking into account that $\Sigma_n \cap \Lambda^n = \Lambda_n$, we see that the inclusion of Σ_n into Σ^n induces the following exact sequence of algebras:

$$0 \to \mathcal{I}_n \to C^\infty(\Sigma^n, \Lambda^n) \to C^\infty(\Sigma_n, \Lambda_n) \to 0,$$

whence $C^\infty(\Sigma_n, \Lambda_n)$ is canonically isomorphic to the quotient algebra $C^\infty(\Sigma^n, \Lambda^n)/\mathcal{I}_n$. Denote

$$\mathcal{E}_n := C^\infty(\Sigma^n, \Lambda^n)/\mathcal{I}_n. \tag{4}$$

Notice that $\mathcal{E}_1 = C^\infty(\Sigma, \Lambda)$.

Remark. A formal neighborhood (X, Y) is the simplest example of a formal manifold which, in general, should be defined as a ringed space on Y. If a formal symplectic groupoid is defined as a formal neighborhood (Σ, Λ), it is too restrictive to require the existence of the manifolds Σ_n for $n \geq 2$. This is why from now on we will automatically replace the algebra $C^\infty(\Sigma_n, \Lambda_n)$ with $\mathcal{E}_n$ for $n \geq 2$. In particular, we consider the comultiplication mapping m^* as a mapping from $C^\infty(\Sigma, \Lambda)$ to $\mathcal{E}_2$ and the algebra morphism ι_n^* as the quotient mapping from $C^\infty(\Sigma^n, \Lambda^n)$ to $\mathcal{E}_n$.

Axioms (A1) and (A2) imply the following identities in the algebra $\mathcal{E}_2$: for $f \in C^\infty(M)$

$$m^*(Sf) = \iota^*(Sf \otimes 1) \tag{5}$$

and

$$m^*(Tf) = \iota^*(1 \otimes Tf), \tag{6}$$

respectively.

In order to state the formal analogues of axioms (A5),(A6),(A8), and (A9) we need one more mapping. Denote by $\delta : \Sigma \to \Sigma \times \Sigma$

the diagonal inclusion of Σ. Since $\delta(\Lambda)$ is the diagonal of $\Lambda \times \Lambda$, the mapping δ induces the dual morphism

$$\delta^* : C^\infty(\Sigma^2, \Lambda^2) \to C^\infty(\Sigma, \Lambda).$$

In what follows $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $\mathcal{E}_2$, i.e., $m^*F = \iota^*G$ in $\mathcal{E}_2$. Axiom (A5) implies that $F(\alpha) = G(\epsilon(s(\alpha)), \alpha)$, whence

$$F = (\delta^* \circ (SE \otimes 1))G. \tag{7}$$

Similarly, it follows from Axiom (A6) that

$$F = (\delta^* \circ (1 \otimes TE))G. \tag{8}$$

Axioms (A8) and (A9) imply that

$$(SE)F = (\delta^* \circ (1 \otimes I))G \text{ and } (TE)F = (\delta^* \circ (I \otimes 1))G, \tag{9}$$

respectively.

Now we need to state the formal analogue of the associativity of the groupoid multiplication. The mapping $m^* \otimes 1$ maps $F \in C^\infty(\Sigma^2, \Lambda^2)$ to a coset in $C^\infty(\Sigma^3, \Lambda^3)$ of the the ideal generated by the functions $Tf \otimes 1 \otimes 1 - 1 \otimes Sf \otimes 1$. This ideal belongs to the ideal $\mathcal{I}_3$. Therefore the image of F with respect to the mapping $m^* \otimes 1$ is a well defined element of $\mathcal{E}_3$. It can be checked using formula (6) that the homomorphism $m^* \otimes 1$ maps the ideal $\mathcal{I}_2$ to $\mathcal{I}_3$. This implies that the mapping $m^* \otimes 1$ induces a well defined mapping from $\mathcal{E}_2$ to $\mathcal{E}_3$, which we will denote $(m_2^1)^*$. Similarly, we construct the mapping $(m_2^2)^* : \mathcal{E}_2 \to \mathcal{E}_3$ induced by $1 \otimes m^*$. The associativity of the groupoid multiplication implies that

$$\left(m_2^1\right)^* \circ m^* = \left(m_2^2\right)^* \circ m^*. \tag{10}$$

To define a formal symplectic groupoid over a Poisson manifold M, we begin with a collection of the following data: a symplectic manifold Σ, a Lagrangian manifold $\Lambda \subset \Sigma$, an embedding $\epsilon : M \to \Sigma$ such that $\epsilon(M) = \Lambda$, its dual $E : C^\infty(\Sigma, \Lambda) \to C^\infty(M)$, a Poisson morphism $S : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ and an anti-Poisson morphism $T : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ such that Sf and Tg Poisson commute for any $f, g \in C^\infty(M)$, and an involutive antisymplectic automorphism $I : C^\infty(\Sigma, \Lambda) \to C^\infty(\Sigma, \Lambda)$. For $f \in C^\infty(M)$ introduce the functions $S_n^k f, T_n^k f \in C^\infty(\Sigma^n, \Lambda^n)$ by formulas (3). For each n define the ideal $\mathcal{I}_n$ in $C^\infty(\Sigma^n, \Lambda^n)$ generated by the functions $S_n^{k+1} f - T_n^k f$, where $f \in C^\infty(M)$ and $1 \le k \le n-1$, and the quotient algebra $\mathcal{E}_n = C^\infty(\Sigma^n, \Lambda^n)/\mathcal{I}_n$ as above. Denote by $\iota_n^* : C^\infty(\Sigma^n, \Lambda^n) \to \mathcal{E}_n$ the quotient mapping. There should exist a comultiplication mapping $m^* : C^\infty(\Sigma, \Lambda) \to \mathcal{E}_2$ which has Property P and satisfies the formal

analogues of axioms (A1) - (A9) given by formulas (5), (6), (1), (7), (8), (2), and (9), respectively. It should generate the mappings

$$\left(m_2^1\right)^*, \left(m_2^2\right)^* : \mathcal{E}_2 \to \mathcal{E}_3$$

as above so that the coassociativity condition (10) is satisfied. In what follows we will refer to the formal analogues of axioms (A1) - (A9) as to axioms (FA1) - (FA9).

3. Formal symplectic realization of a Poisson manifold

If Σ is a symplectic manifold, M a Poisson manifold, $s : \Sigma \to M$ a surjective submersion which is a Poisson mapping, and $\epsilon : M \to \Sigma$ an embedding such that $s \circ \epsilon = \mathrm{id}_M$ and $\Lambda = \epsilon(M) \subset \Sigma$ is a Lagrangian manifold, then Σ is called a strict symplectic realization of the Poisson manifold M (see [6]). It is known that, given a strict symplectic realization Σ of the Poisson manifold M, there exists a canonical local symplectic groupoid over the manifold M defined on a neighborhood of Λ in Σ, such that s is its source mapping ([6], Thm. 1.2 on page 44).

In this section we will prove a formal version of this theorem. Let Σ be a symplectic manifold, M a Poisson manifold, and $\epsilon : M \to \Sigma$ an embedding such that $s \circ \epsilon = \mathrm{id}_M$ and $\Lambda = \epsilon(M) \subset \Sigma$ is a Lagrangian manifold. Denote by $E : C^\infty(\Sigma, \Lambda) \to C^\infty(M)$ the dual mapping of ϵ. Then, if there is given a formal Poisson morphism $S : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ such that $ES = \mathrm{id}_{C^\infty(M)}$, we say that the formal neighborhood (Σ, Λ) is a formal strict symplectic realization of the Poisson manifold M.

Theorem 1. *Given a formal strict symplectic realization of a Poisson manifold M on the formal neighborhood (Σ, Λ) of a Lagrangian submanifold Λ of a symplectic manifold Σ via a formal Poisson morphism $S : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$, there exists a unique formal symplectic groupoid on (Σ, Λ) over the manifold M such that S is its source mapping.*

Assume there is a formal strict symplectic realization (Σ, Λ) of the Poisson manifold M. Given an element $F \in C^\infty(\Sigma, \Lambda)$, denote by $H_F = \{F, \cdot\}_\Sigma$ the formal Hamiltonian vector field corresponding to the formal Hamiltonian F. Denote by λ the representation of the Lie algebra $\mathfrak{g} := (C^\infty(M), \{\cdot, \cdot\}_M)$ on the space $C^\infty(\Sigma, \Lambda)$ given by the formula

$$\lambda(f) = H_{Sf},$$

where $f \in \mathfrak{g}$. Extend the representation λ to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ and define a mapping

$$\langle F \rangle : \mathcal{U}(\mathfrak{g}) \to C^\infty(M)$$

by the formula

$$\langle F \rangle(u) = E(\lambda(u)F),$$

where $u \in \mathcal{U}(\mathfrak{g})$. Denote the multiplication in the algebra $\mathcal{U}(\mathfrak{g})$ by $\bullet$, so that $f \bullet g - g \bullet f = \{f, g\}_M$ for $f, g \in \mathfrak{g}$. We will often work in the following local framework on Σ. Let U be a Darboux chart on Σ with the local coordinates $\{x^k, \xi_l\}$ such that $\Lambda \cap U$ is given by the equations $\xi = 0$ and for $F, G \in C^\infty(U)$

$$\{F, G\}_\Sigma = \partial^k F \, \partial_k G - \partial^k G \, \partial_k F, \tag{11}$$

where $\partial_k = \partial/\partial x^k$ and $\partial^l = \partial/\partial \xi_l$. We will say that these Darboux coordinates are *standard*. In this framework a formal function $F \in C^\infty(\Sigma, \Lambda)$ will be represented on the formal neighborhood $(U, \Lambda \cap U)$ as an element of $C^\infty(\Lambda \cap U)[[\xi]]$ and the coordinates ξ_l will be treated as formal variables. We identify M and Λ via the mapping ϵ, so that $\{x^k\}$ are used also as coordinates on $\epsilon^{-1}(\Lambda \cap U)$. In particular, for $F = F(x, \xi)$ we have $E(F)(x) = F(x, 0)$. A local expression for the Poisson bracket on M is

$$\{f, g\}_M = \eta^{ij} \partial_i f \, \partial_j g, \tag{12}$$

where $f, g \in C^\infty(M)$.

Lemma 2. *Given a function $f \in C^\infty(M)$, the element $Sf \in C^\infty(\Sigma, \Lambda)$ can be written in standard local coordinates (x, ξ) on a Darboux chart $U \subset \Sigma$ as*

$$Sf(x, \xi) = f(x) + \alpha^{ij}(x) \, \partial_i f \, \xi_j \pmod{\xi^2} \tag{13}$$

for some function $\alpha^{ij}(x)$ such that $\alpha^{ij} - \alpha^{ji} = \eta^{ij}$.

Proof. Denote $s^i = Sx^i$. Since $E(s^i) = x^i$, expanding $s^i(x, \xi)$ with respect to the formal variables ξ we get that $s^i = x^i + \alpha^{ij}(x)\xi_j \pmod{\xi^2}$ for some function $\alpha^{ij}(x)$. It follows from the fact that S is an algebra morphism, that

$$Sf(x, \xi) = f(s(x, \xi)) = f(x) + \alpha^{ij}(x) \, \partial_i f \, \xi_j \pmod{\xi^2}.$$

Notice that the 'substitution' $f(s(x, \xi))$ is understood as a composition of formal series. Since S is a Poisson morphism, we have that $\{Sf, Sg\}_\Sigma = S(\{f, g\}_M)$ for any $f, g \in C^\infty(M)$. On the one hand, according to formulas (11) and (13),

$$\partial^k(Sf) \, \partial_k(Sg) - \partial^k(Sf) \, \partial_k(Sg) = \alpha^{ik} \, \partial_i f \, \partial_k g - \alpha^{ik} \, \partial_i g \, \partial_k f \pmod{\xi}.$$

On the other hand, $S(\{f,g\}_M) = \eta^{ij}\,\partial_i f\,\partial_j g \pmod{\xi}$, which concludes the proof. □

Lemma 3. *For any $F \in C^\infty(\Sigma, \Lambda)$ and $u \in \mathcal{U}(\mathfrak{g})$ the mapping $C^\infty(M) \ni f \mapsto \langle F\rangle(f \bullet u)$ is a derivation on $C^\infty(M)$.*

Proof. Let us show that the mapping $C^\infty(M) \ni f \mapsto E(H_{Sf}F)$ is a derivation. Using Lemma 2 and formula (13), we obtain that in local Darboux coordinates

$$\begin{aligned} E(H_{Sf}F) = E(\{Sf, F\}_\Sigma) = E(\partial^k(Sf)\partial_k F - \partial^k F \partial_k(Sf)) = \\ \alpha^{ik}\partial_i f\, E(\partial_k F) - \partial_k f\, E(\partial^k F). \end{aligned} \tag{14}$$

To prove the statement of the Lemma, the element F should be replaced with $\lambda(u)F$. □

Denote by $\mathcal{C}$ the space of linear mappings $C : \mathcal{U}(\mathfrak{g}) \to C^\infty(M)$ such that for any $u \in \mathcal{U}(\mathfrak{g})$ the mapping $C^\infty(M) \ni f \mapsto C(f \bullet u)$ is a derivation on $C^\infty(M)$. Lemma 3 implies that the mapping

$$\chi : F \mapsto \langle F\rangle$$

maps $C^\infty(\Sigma, \Lambda)$ to $\mathcal{C}$. We will prove that the mapping $\chi : C^\infty(\Sigma, \Lambda) \to \mathcal{C}$ is actually a bijection. Each element $C \in \mathcal{C}$ is completely determined by the family of polydifferential operators $\{C_n\},\ n \geq 0$, on M, where C_n is the n-differential operator such that

$$C_n(f_1, \ldots, f_n) = C(f_1 \bullet \ldots \bullet f_n). \tag{15}$$

The operators $\{C_n\}$ enjoy the following two properties.
Property A. Each operator $C_n, n \geq 0$, is a derivation in the first argument.
Property B. For any k, n such that $1 \leq k \leq n-1$

$$\begin{aligned} C_n(f_1, \ldots, f_k, f_{k+1}, \ldots f_n) - C_n(f_1, \ldots, f_{k+1}, f_k, \ldots f_n) = \\ C_{n-1}(f_1, \ldots, \{f_k, f_{k+1}\}_M, \ldots f_n). \end{aligned} \tag{16}$$

We will call a family $\{C_n(f_1, \ldots, f_n)\}, n \geq 0$, of polydifferential operators on M *coherent* if it has Properties A and B. The correspondence $C \mapsto \{C_n\}$ given by formula (15) is a bijection between the space $\mathcal{C}$ and the set of all coherent families.

It is easy to show that each operator C_n from a coherent family annihilates constants (i.e., $C_n(f_1, \ldots, f_n) = 0$ if $f_k = 1$ for at least one index k) and is of order not greater than k in the kth argument for $1 \leq k \leq n$.

It is important to notice that if $\{C_n\}, n \geq 0$, is a coherent family on M and $\phi \in C^\infty(M)$, then the operators $\{\phi \cdot C_n\}, n \geq 0$, also form a

coherent family. This observation means that one can apply partition of unity arguments to the coherent families.

The standard increasing filtration on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ induces the dual decreasing filtration $\{\mathcal{C}^{(n)}\}$ on $\mathcal{C}$, i.e., $\mathcal{C}^{(n)}$ consists of all operators C such that the corresponding coherent family $\{C_k\}$ satisfies the condition $C_k = 0$ for $0 \leq k \leq n-1$. The following lemma is an immediate consequence of Properties A and B of the coherent families.

Lemma 4. *If $C \in \mathcal{C}^{(n)}$, then $C_n(f_1, \ldots, f_n)$ is a symmetric multiderivation on M (i.e., of order one in each argument and null on constants).*

We will also consider finite coherent families $\{C_k\}, 0 \leq k \leq n$. It turns out that any n-element coherent family can be extended to an $(n+1)$-element coherent family.

Theorem 2. *Any n-element coherent family $\{C_k\}, 0 \leq k \leq n-1$, can be extended to an $(n+1)$-element coherent family $\{C_k\}, 0 \leq k \leq n$. The operator C_n is unique up to an arbitrary symmetric multiderivation.*

We will prove Theorem 2 in the Appendix.

Given an element $F \in C^\infty(\Sigma, \Lambda)$, set $C = \langle F \rangle = \chi(F)$. We will denote the corresponding operator C_n by $\langle F \rangle_n$, so that

$$\langle F \rangle_n(f_1, \ldots, f_n) = E(H_{Sf_1} \ldots H_{Sf_n} F),$$

where $f_i \in C^\infty(M)$. Denote by $\mathcal{J} = I_\Lambda / I_\Lambda^\infty$ the kernel of the mapping $E : C^\infty(\Sigma, \Lambda) \to C^\infty(M)$, i.e., the ideal of formal functions on (Σ, Λ) vanishing on Λ. The powers of this ideal, $\{\mathcal{J}^n\}$, form a decreasing filtration on the algebra $C^\infty(\Sigma, \Lambda)$. Consider an element $F \in \mathcal{J}^n$. The operator $\langle F \rangle_n$ vanishes if $k < n$, therefore $\langle F \rangle \in \mathcal{C}^{(n)}$. Thus the mapping $\chi : C^\infty(\Sigma, \Lambda) \to \mathcal{C}$ is a morphism of filtered spaces. Notice that the filtrations on $C^\infty(\Sigma, \Lambda)$ and $\mathcal{C}$ are complete and separated.

Lemma 5. *Let F be an arbitrary element in $\mathcal{J}^n$. Then*

$$\langle F \rangle_n(f_1, \ldots, f_n) = E(H_{Sf_1} \ldots H_{Sf_n} F),$$

$f_i \in C^\infty(M)$, is a symmetric multiderivation which does not depend on the choice of the source mapping S. The mapping $\chi : F \mapsto \langle F \rangle_n$ induces an isomorphism of $\mathcal{J}^n / \mathcal{J}^{n+1}$ onto the space of symmetric n-derivations on M.

Proof. In standard local Darboux coordinates (x^k, ξ_l) on Σ a function $F \in \mathcal{J}^n$ can be written as $F(x, \xi) = F^{i_1 \ldots i_n}(x) \xi_{i_1} \ldots \xi_{i_n} \pmod{\xi^{n+1}}$, where $F^{i_1 \ldots i_n}(x)$ is symmetric in $i_1, \ldots, i_n$. Taking into account formula (11) and that $Sf = f \pmod{\xi}$, we get that

$$\langle F \rangle_n(f_1, \ldots, f_n) = E(H_{Sf_1} \ldots H_{Sf_n} F) = (-1)^n n! \, F^{i_1 \ldots i_n} \partial_{i_1} f_1 \ldots \partial_{i_n} f_n.$$

This calculation shows that $F^{i_1\dots i_n}(x)$ is a symmetric tensor which does not depend on the choice of the source mapping S and that the mapping $\chi : F \mapsto \langle F\rangle_n$ induces an isomorphism of $\mathcal{J}^n/\mathcal{J}^{n+1}$ onto the space of symmetric n-derivations on M. $\square$

Remark. One can describe the tensor $F^{i_1\dots i_n}(x)$ (and the corresponding multiderivation) independently, regardless the existence of the source mapping S. The description is based upon the identification of the conormal bundle of $\Lambda \subset \Sigma$ with its tangent bundle $T\Lambda$.

Proposition 1. *The mapping $\chi : C^\infty(\Sigma, \Lambda) \to \mathcal{C}$ is a bijection.*

Proof. The mapping χ is a morphism of complete Hausdorff filtered spaces. According to Theorem 2, the quotient space $\mathcal{C}^{(n)}/\mathcal{C}^{(n+1)}$ is isomorphic to the space of symmetric n-derivations on M, the isomorphism being induced by the mapping $\mathcal{C}^{(n)} \ni C \mapsto C_n$. Lemma 5 thus shows that the mapping χ induces an isomorphism of $\mathcal{J}^n/\mathcal{J}^{n+1}$ onto $\mathcal{C}^{(n)}/\mathcal{C}^{(n+1)}$, whence the Proposition follows. $\square$

Using Proposition 1 we will transfer the structure of Poisson algebra from $C^\infty(\Sigma, \Lambda)$ to $\mathcal{C}$ via the mapping χ. It turns out that the resulting Poisson algebra structure on $\mathcal{C}$ does not depend on the mapping S and can be described canonically and intrinsically in terms of the Poisson structure on M.

Denote by $\delta_{\mathcal{U}} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ the standard cocommutative coproduct on $\mathcal{U}(\mathfrak{g})$, so that $\delta_{\mathcal{U}}(f) = f \otimes 1 + 1 \otimes f$ for $f \in \mathfrak{g}$.

For $F, G \in C^\infty(\Sigma, \Lambda),\ u \in \mathcal{U}(\mathfrak{g})$, we have

$$\langle FG\rangle(u) = E(\lambda(u)(FG)) = \sum_i E((\lambda(u_i')F)(\lambda(u_i'')G)) =$$
$$\sum_i E(\lambda(u_i')F)E(\lambda(u_i'')G) = \sum_i \langle F\rangle(u_i')\langle G\rangle(u_i''). \tag{17}$$

Here, as well as in the rest of the paper, we use the notation

$$\delta_{\mathcal{U}}(u) = \sum_i u_i' \otimes u_i''.$$

For $A, B \in \mathcal{C}$ denote by AB their convolution product, so that

$$(AB)(u) = \sum_i A(u_i')B(u_i''). \tag{18}$$

This product is commutative since $\delta_{\mathcal{U}}$ is cocommutative. Formula (17) shows that the mapping χ is an algebra isomorphism from $C^\infty(\Sigma, \Lambda)$ to $\mathcal{C}$ endowed with the convolution product.

For $F, G \in C^\infty(\Sigma, \Lambda)$ we obtain by setting $f = E(G)$ in formula (14) that

$$E\left(H_{(SE)(G)}F\right) = \alpha^{ik} E(\partial_i G) E(\partial_k F) - E(\partial^k F) E(\partial_k G). \tag{19}$$

Swapping F and G in (19) and subtracting the resulting equation from (19) we get, taking into account formulas (11), (12), and Lemma 2, that

$$\begin{aligned} E(\{F, G\}_\Sigma) = E\left(H_{(SE)(F)}G\right) - E\left(H_{(SE)(G)}F\right) - \\ \{E(F), E(G)\}_M. \end{aligned} \tag{20}$$

For $u, v \in \mathcal{U}(\mathfrak{g})$

$$\begin{aligned} E\left(H_{(SE)(\lambda(u)F)}\lambda(v)G\right) = E\left(H_{S(\langle F\rangle(u))}\lambda(v)G\right) = E\left(\lambda(\langle F\rangle(u))\lambda(v)G\right) = \\ E\left(\lambda(\langle F\rangle(u) \bullet v)G\right) = \langle G\rangle(\langle F\rangle(u) \bullet v). \end{aligned} \tag{21}$$

Using the Jacobi identity, formulas (20) and (21), we obtain that

$$\begin{aligned} \langle\{F, G\}_\Sigma\rangle(u) = E\left(\lambda(u)\{F, G\}_\Sigma\right) = \\ \sum_i E\left(\{\lambda(u_i')F, \lambda(u_i'')G\}_\Sigma\right) = \sum_i \big(\langle G\rangle(\langle F\rangle(u_i') \bullet u_i'') - \\ \langle F\rangle(\langle G\rangle(u_i'') \bullet u_i') - \{\langle F\rangle(u_i'), \langle G\rangle(u_i'')\}_M\big). \end{aligned} \tag{22}$$

Notice that in formula (22) the functions $\langle F\rangle(u_i'), \langle G\rangle(u_i'') \in C^\infty(M)$ are used as elements of the Lie algebra $\mathfrak{g}$. Formula (22) shows that the mapping χ transfers the Poisson bracket from $C^\infty(\Sigma, \Lambda)$ to the following Poisson bracket on $\mathcal{C}$:

$$\begin{aligned} \{A, B\}_{\mathcal{C}}(u) = \sum_i \Big(B(A(u_i') \bullet u_i'') - A(B(u_i'') \bullet u_i') - \\ \{A(u_i'), B(u_i'')\}_M\Big), \end{aligned} \tag{23}$$

where $A, B \in \mathcal{C}$. We see that the bracket (23) is defined intrinsically in terms of the Poisson structure on M. One can prove that the bracket (23) defines a Poisson algebra structure on the algebra $\mathcal{C}$ regardless the existence of the mapping S.

Now we can construct an anti-Poisson morphism $T : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ such that $ET = \mathrm{id}_{C^\infty(M)}$ and the formal functions Sf and Tg Poisson commute for any $f, g \in C^\infty(M)$.

Denote by $\epsilon_{\mathcal{U}} : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ the counit mapping of the algebra $\mathcal{U}(\mathfrak{g})$, so that $\epsilon_{\mathcal{U}}(\mathbf{1}) = 1$ and $\epsilon_{\mathcal{U}}(f) = 0$ for $f \in \mathfrak{g}$. Here $\mathbf{1}$ is the unity in the algebra $\mathcal{U}(\mathfrak{g})$. Let $\mathbf{k}$ denote the trivial representation of the algebra $\mathcal{U}(\mathfrak{g})$ on $C^\infty(M)$, i.e., such that

$$\mathbf{k}(u)f = \epsilon_{\mathcal{U}}(u) \cdot f$$

for $u \in \mathcal{U}(\mathfrak{g})$, $f \in C^\infty(M)$. For $f \in C^\infty(M)$ consider a mapping $X_f \in \mathcal{C}$ such that

$$X_f(u) = \mathbf{k}(u)f,$$

where $u \in \mathcal{U}(\mathfrak{g})$. For $f, g \in C^\infty(M)$ and $u \in \mathcal{U}(\mathfrak{g})$ we get from formula (23):

$$\{X_f, X_g\}_{\mathcal{C}}(u) = -\left(\sum_i \epsilon_{\mathcal{U}}(u_i') \cdot \epsilon_{\mathcal{U}}(u_i'')\right)\{f, g\}_M = \\ -\epsilon_{\mathcal{U}}(u)\{f, g\}_M = -X_{\{f,g\}_M}(u).$$

Thus the mapping $f \mapsto X_f$ is an anti-Poisson morphism from $C^\infty(M)$ to $\mathcal{C}$.

Let $\mathbf{h}$ denote the representation of the Lie algebra $\mathfrak{g}$ on $C^\infty(M)$ by the Hamiltonian vector fields, $\mathbf{h}(f) = \{f, \cdot\}_M$, $f \in \mathfrak{g}$. Extend it to $\mathcal{U}(\mathfrak{g})$. It follows from the fact that $ES = \mathrm{id}_{C^\infty(M)}$, that

$$\langle Sf\rangle(u) = \mathbf{h}(u)f. \tag{24}$$

For $f, g \in C^\infty(M)$ we get from formula (23) that $\langle Sf\rangle$ Poisson commutes with X_g:

$$\{\langle Sf\rangle, X_g\}_{\mathcal{C}}(u) = \sum_i \left(-\epsilon_{\mathcal{U}}(u_i'')\mathbf{h}(g \bullet u_i')f - \epsilon_{\mathcal{U}}(u_i'')\{\mathbf{h}(u_i')f, g\}_M\right) = \\ \sum_i \left(-\epsilon_{\mathcal{U}}(u_i'')\mathbf{h}(g)\mathbf{h}(u_i')f + \epsilon_{\mathcal{U}}(u_i'')\{g, \mathbf{h}(u_i')f\}_M\right) = 0.$$

Taking into account that the mapping χ is a Poisson algebra isomorphism of $C^\infty(\Sigma, \Lambda)$ onto $\mathcal{C}$, we define the mapping $T : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ as follows. For $f \in C^\infty(M)$ Tf is chosen to be the unique element of $C^\infty(\Sigma, \Lambda)$ such that

$$\langle Tf\rangle = X_f. \tag{25}$$

We see that the mapping $T : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ is an anti-Poisson morphism and for any $f, g \in C^\infty(M)$ the formal functions Sf and Tg Poisson commute. Thus the mapping T enjoys the properties of the target mapping. On the other hand, if T is the target mapping of a formal symplectic groupoid on (Σ, Λ) whose source mapping is S, it is straightforward that

$$\langle Tf\rangle(u) = \mathbf{k}(u)f.$$

It means that the target mapping T is uniquely determined by the source mapping S.

In order to construct the inverse mapping I and the comultiplication m^* from the mappings S and T, we will consider mappings from tensor powers of $\mathcal{U}(\mathfrak{g})$ to $C^\infty(M)$ which generalize the mappings from $\mathcal{C}$. The

space $\mathrm{Hom}(\mathcal{U}(\mathfrak{g})^{\otimes n}, C^\infty(M))$ is endowed with the convolution product defined on its elements A, B as follows:

$$(AB)(u_1 \otimes \ldots \otimes u_n) = \sum_{i_1,\ldots,i_n} A(u'_{1i_1} \otimes \ldots \otimes u'_{ni_n}) B(u''_{1i_1} \otimes \ldots \otimes u''_{ni_n}), \tag{26}$$

where

$$\delta_{\mathcal{U}}(u_k) = \sum_i u'_{ki} \otimes u''_{ki}.$$

Denote by $\{\cdot,\cdot\}_{\Sigma^n}$ the Poisson bracket on $C^\infty(\Sigma^n)$ (and on $C^\infty(\Sigma^n, \Lambda^n)$) corresponding to the product Poisson structure. For $F \in C^\infty(\Sigma^n, \Lambda^n)$ let $H_F = \{F, \cdot\}_{\Sigma^n}$ denote the corresponding formal Hamiltonian vector field on (Σ^n, Λ^n). Introduce representations $\lambda_n^k,\ 0 \leq k \leq n$, of the Lie algebra $\mathfrak{g}$ on $C^\infty(\Sigma^n, \Lambda^n)$ by the following formulas:

$$\lambda_n^0(f) = H_{S_n^1 f},\ \lambda_n^n(f) = -H_{T_n^n f},\ \text{and}\ \lambda_n^k(f) = H_{(S_n^{k+1} f - T_n^k f)}$$

for $1 \leq k \leq n-1$, where the functions $S_n^k, T_n^k \in C^\infty(\Sigma^n, \Lambda^n)$ are given by formulas (3). These representations pairwise commute. Notice that in these notations the representation λ is denoted λ_1^0. Denote the representation λ_1^1 by ρ so that $\rho(f) = -H_{Tf}$ for $f \in \mathfrak{g}$. Extend the representations λ_n^k to the algebra $\mathcal{U}(\mathfrak{g})$. For $u \in \mathcal{U}(\mathfrak{g})$

$$\lambda_n^0(u) = \underbrace{\lambda(u) \otimes 1 \otimes \ldots \otimes 1}_{n},\ \lambda_n^n(u) = \underbrace{1 \otimes \ldots \otimes 1 \otimes \rho(u)}_{n},$$

$$\text{and} \quad \lambda_n^k(u) = \sum_i \underbrace{1 \otimes \ldots \otimes \overbrace{\rho(u'_i)}^{k-\text{th}} \otimes \lambda(u''_i) \otimes \ldots \otimes 1}_{n}, \tag{27}$$

where $1 \leq k \leq n-1$. Let $\epsilon_n : M \to \Sigma^n$ denote the composition of the identification mapping from M onto Λ_n with the inclusion of Λ_n into Σ^n. Since $\Lambda_n \subset \Lambda^n$, the mapping ϵ_n induces the algebra morphism $E_n : C^\infty(\Sigma^n, \Lambda^n) \to C^\infty(M)$. In particular, $\epsilon = \epsilon_1$ and $E = E_1$. After some preparations we will show that the morphism E_n intertwines the representations $\mathbf{h}$ and $\sum_{k=0}^n \lambda_n^k$.

We cover the submanifold $\Lambda^n \subset \Sigma^n$ by Cartesian products $U_1 \times \ldots \times U_n$ of standard Darboux charts $U_i \subset \Sigma$ and use the coordinates $\{x^i_{[k]}, \xi_{j[k]}\}$ on the k-th factor. In particular, in local coordinates $S_n^k f = (Sf)(x_{[k]}, \xi_{[k]})$ and $T_n^k f = (Tf)(x_{[k]}, \xi_{[k]})$. For a function $F = F(x_{[1]}, \xi_{[1]}, \ldots, x_{[n]}, \xi_{[n]})$ on $U_1 \times \ldots \times U_n$ we have $E_n(F) =$

$F(x, 0, \ldots, x, 0)$. We will use below the following obvious formulas,

$$E_n(f(x_{[k]})F) = f(x)E_n(F) \text{ and } \partial_i E_n(F) = \sum_{k=1}^{n} E_n(\partial_{i[k]}F), \tag{28}$$

where $\partial_{i[k]} = \partial/\partial x^i_{[k]}$. It can be proved as in Lemma 2 that in local Darboux coordinates (x, ξ)

$$Tf(x, \xi) = f(x) + \alpha^{ji}(x)\,\partial_i f\,\xi_j \pmod{\xi^2}, \tag{29}$$

where the function $\alpha^{ij}(x)$ is the same as in formula (13).

Lemma 6. *For $f \in C^\infty(M)$ and $F \in C^\infty(\Sigma^n, \Lambda^n)$*

$$\mathbf{h}(f)E_n(F) = \sum_{k=0}^{n} E_n(\lambda_n^k(f)F).$$

Proof. Using formulas (11),(12), (13),(29), and Lemma 2 we get:

$$E(H_{(Sf-Tf)}F) = E(\{Sf - Tf, F\}_\Sigma) = \\ E(\partial^k(\eta^{ij}\partial_i f\xi_j)\partial_k F) = \mathbf{h}(h)E(F).$$

Now the Lemma follows from formulas (28) and the fact that

$$\sum_{k=0}^{n} \lambda_n^k(f) = \sum_{k=1}^{n} H_{(S_n^k f - T_n^k f)}.$$

□

Denote by $\mathcal{C}_n$ the subspace of $\mathrm{Hom}(\mathcal{U}(\mathfrak{g})^{\otimes(n+1)}, C^\infty(M))$ of the mappings C such that

- for any $C \in \mathcal{C}_n,\ u_i \in \mathcal{U}(\mathfrak{g}), 0 \le i \le n$, and k satisfying $0 \le k \le n$, the mapping

  $$C^\infty(M) \ni f \mapsto C(u_0 \otimes \ldots \otimes f \bullet u_k \otimes \ldots \otimes u_n)$$

  is a derivation on $C^\infty(M)$; and
- for any $f \in C^\infty(M)$

$$\mathbf{h}(f)C(u_0 \otimes \ldots \otimes u_n) = \sum_{k=0}^{n} C(u_0 \otimes \ldots \otimes f \bullet u_k \otimes \ldots \otimes u_n). \tag{30}$$

The space $\mathcal{C}_n$ is closed under the convolution product and thus is an algebra. For an element $F \in C^\infty(\Sigma^n, \Lambda^n)$ define a mapping

$$\langle\langle F\rangle\rangle \in \mathrm{Hom}(\mathcal{U}(\mathfrak{g})^{\otimes(n+1)}, C^\infty(M))$$

such that

$$\langle\langle F\rangle\rangle(u_0 \otimes \ldots \otimes u_n) = E_n(\lambda_n^0(u_0)\ldots\lambda_n^n(u_n)F). \tag{31}$$

A straightforward generalization of the proof of Lemma 3 shows that for each k satisfying $0 \leq k \leq n$ the mapping

$$C^\infty(M) \ni f \mapsto \langle\langle F \rangle\rangle (u_0 \otimes \ldots \otimes f \bullet u_k \otimes \ldots \otimes u_n)$$

is a derivation on $C^\infty(M)$. It follows from Lemma 6 that the mapping $C = \langle\langle F \rangle\rangle$ satisfies formula (30). Thus the mapping

$$C^\infty(\Sigma^n, \Lambda^n) \ni F \mapsto \langle\langle F \rangle\rangle$$

maps $C^\infty(\Sigma^n, \Lambda^n)$ to $\mathcal{C}_n$. Denote this mapping by χ_n. A simple calculation shows that $\chi_n : C^\infty(\Sigma^n, \Lambda^n) \to \mathcal{C}_n$ is an algebra homomorphism.

Denote by $\tilde{\mathcal{C}}_n$ the subspace of $\mathrm{Hom}(\mathcal{U}(\mathfrak{g})^{\otimes n}, C^\infty(M))$ consisting of the elements $C \in \tilde{\mathcal{C}}_n$ such that for any $u_i \in \mathcal{U}(\mathfrak{g}), 1 \leq i \leq n$, and k satisfying $1 \leq k \leq n$, the mapping

$$C^\infty(M) \ni f \mapsto C(u_1 \otimes \ldots \otimes f \bullet u_k \otimes \ldots \otimes u_n)$$

is a derivation on $C^\infty(M)$. Notice that in these notations $\mathcal{C} = \tilde{\mathcal{C}}_1$. The space $\tilde{\mathcal{C}}_n$ is also an algebra with respect to the convolution product.

Consider a reduction mapping $C \mapsto \tilde{C}$ from $\mathcal{C}_n$ to $\tilde{\mathcal{C}}_n$ defined as follows:

$$\tilde{C}(u_1 \otimes \ldots \otimes u_n) = C(u_1 \otimes \ldots \otimes u_n \otimes \mathbf{1}),$$

where $\mathbf{1}$ is the unity in the algebra $\mathcal{U}(\mathfrak{g})$ (which should not be confused with the unit constant $1 \in \mathfrak{g}$). Formula (30) implies that the reduction mapping $C \mapsto \tilde{C}$ is a bijection of $\mathcal{C}_n$ onto $\tilde{\mathcal{C}}_n$. It is easy to check that the reduction mapping $C \mapsto \tilde{C}$ is an algebra isomorphism of $\mathcal{C}_n$ onto $\tilde{\mathcal{C}}_n$. A straightforward calculation shows that the reduction mapping pulls back the Poisson bracket (23) on $\mathcal{C} = \tilde{\mathcal{C}}_1$ to the Poisson bracket $\{\cdot,\cdot\}_{\mathcal{C}_1}$ on $\mathcal{C}_1$ defined as follows. For $A, B \in \mathcal{C}_1$ and $u, v \in \mathcal{U}(\mathfrak{g})$

$$\{A, B\}_{\mathcal{C}_1}(u \otimes v) = -\sum_{i,j} \Big(A\big((B(u_i'' \otimes v_j'') \bullet u_i') \otimes v_j'\big) + B\big(u_i'' \otimes (A(u_i' \otimes v_j') \bullet v_j'')\big)\Big), \tag{32}$$

where

$$\delta_{\mathcal{U}}(u) = \sum_i u_i' \otimes u_i'' \text{ and } \delta_{\mathcal{U}}(v) = \sum_j v_j' \otimes v_j''.$$

Recall that in (32) the functions $A(u_i' \otimes v_j'), B(u_i'' \otimes v_j'') \in C^\infty(M)$ are treated as elements of the Lie algebra $\mathfrak{g}$. The right-hand side of formula (32) is skew-symmetric due to formula (30) and cocommutativity of the coproduct $\delta_{\mathcal{U}}$.

For $F \in C^\infty(\Sigma, \Lambda)$ the mapping $\langle\langle F \rangle\rangle \in \mathcal{C}_1$ such that

$$\langle\langle F \rangle\rangle (u \otimes v) = E(\lambda(u)\rho(v)F)$$

for $u, v \in \mathcal{U}(\mathfrak{g})$ is completely determined by its reduction $\langle F\rangle(u) = E(\lambda(u)F)$. Thus the mapping $\chi_1 : C^\infty(\Sigma,\Lambda) \to \mathcal{C}_1$ is a Poisson algebra isomorphism. This isomorphism will be used to introduce the inverse mapping I on $C^\infty(\Sigma,\Lambda)$ in the most transparent way.

A simple calculation shows that for $f \in C^\infty(M)$ and $u, v \in \mathcal{U}(\mathfrak{g})$

$$\langle\langle Sf\rangle\rangle(u\otimes v) = \mathbf{h}(u)\mathbf{k}(v)f \text{ and } \langle\langle Tf\rangle\rangle(u\otimes v) = \mathbf{h}(v)\mathbf{k}(u)f. \tag{33}$$

Given a mapping $C : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \to C^\infty(M)$, denote by $C^\dagger$ the mapping from $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ to $C^\infty(M)$ such that

$$C^\dagger(u \otimes v) = C(v \otimes u)$$

for $u, v \in \mathcal{U}(\mathfrak{g})$. It is easy to check that the mapping $C \mapsto C^\dagger$ leaves invariant the space $\mathcal{C}_1$. Formulas (33) indicate that

$$\langle\langle Sf\rangle\rangle^\dagger = \langle\langle Tf\rangle\rangle.$$

Using formulas (26) for $n = 2$ and (32) one can readily show that the mapping $C \mapsto C^\dagger$ induces an involutive anti-Poisson automorphism of the Poisson algebra $\mathcal{C}_1$. Define a unique mapping I on $C^\infty(\Sigma,\Lambda)$ such that for $F \in C^\infty(\Sigma,\Lambda)$ and $u, v \in \mathcal{U}(\mathfrak{g})$

$$\langle\langle I(F)\rangle\rangle(u\otimes v) = \langle\langle F\rangle\rangle(v\otimes u). \tag{34}$$

It follows that the mapping I is an involutive anti-Poisson automorphism of $C^\infty(\Sigma,\Lambda)$ such that

$$IS = T \text{ and } IT = S. \tag{35}$$

Now assume that I is the inverse mapping of a formal symplectic groupoid on (Σ,Λ) over M with the source mapping S (and target mapping T). Then for $f \in C^\infty(M)$ and $F \in C^\infty(\Sigma,\Lambda)$

$$I(\lambda(f)F) = I\left(\{Sf, F\}_\Sigma\right) = -\{ISf, I(F)\}_\Sigma = \\ -\{Tf, I(F)\}_\Sigma = \rho(f)I(F).$$

Therefore $I \circ \lambda(u) = \rho(u) \circ I$ for $u \in \mathcal{U}(\mathfrak{g})$. Since I is involutive, $I \circ \rho(u) = \lambda(u) \circ I$. One can derive from the groupoid axioms that $i \circ \epsilon = \epsilon$. Similarly, for a formal symplectic groupoid, the formula

$$(EI)(F) = E(F),$$

where $F \in C^\infty(\Sigma,\Lambda)$, holds. Now,

$$\langle\langle I(F)\rangle\rangle(u\otimes v) = E(\lambda(u)\rho(v)I(F)) = E(I(\rho(u)\lambda(v)F)) = \\ E(\rho(u)\lambda(v)F) = E(\lambda(v)\rho(u)F) = \langle\langle F\rangle\rangle(v\otimes u),$$

which means that the inverse mapping I is uniquely determined by the source mapping S.

Our next task is to construct the comultiplication of the formal symplectic groupoid from the source and target mappings. Denote by $\mathcal{I}_n$, as above, the ideal in $C^\infty(\Sigma^n, \Lambda^n)$ generated by the functions

$$S_n^{k+1} f - T_n^k f, \tag{36}$$

where $f \in C^\infty(M),\ 1 \leq k \leq n-1$, and set $\mathcal{E}_n = C^\infty(\Sigma^n, \Lambda^n)/\mathcal{I}_n$ as in formula (4).

Lemma 7. *The representations λ_n^k leave invariant the ideal $\mathcal{I}_n$. The ideal $\mathcal{I}_n$ is in the kernel of the algebra morphism $\chi_n : C^\infty(\Sigma^n, \Lambda^n) \to \mathcal{C}_n$.*

Proof. For $f, g \in C^\infty(M)$

$$\lambda_n^k(f)(S_n^{l+1} - T_n^l)g = \begin{cases} (S_n^{k+1} - T_n^k)\{f, g\}_M & \text{if } k = l \\ 0 & \text{otherwise,} \end{cases}$$

whence we see that the representations λ_n^k leave invariant the ideal $\mathcal{I}_n$. Since $E_n(S_n^k f) = f$ and $E_n(T_n^k f) = f$, we get that $E_n(S_n^{k+1} f - T_n^k f) = 0$. Therefore the ideal $\mathcal{I}_n$ is in the kernel of the algebra morphism $E_n : C^\infty(\Sigma^n, \Lambda^n) \to C^\infty(M)$. Now the Lemma follows from formula (31). $\square$

Lemma 7 implies that the homomorphism χ_n factors through $\mathcal{E}_n$. Denote by ψ_n the induced homomorphism from $\mathcal{E}_n$ to $\mathcal{C}_n$. Notice that $\mathcal{E}_1 = C^\infty(\Sigma, \Lambda)$ and $\psi_1 = \chi_1$. It can be obtained by a straightforward generalization of the proof of Proposition 1 that the induced homomorphism $\psi_n : \mathcal{E}_n \to \mathcal{C}_n$ is, actually, an isomorphism. Introduce a mapping $\theta : \mathcal{C}_1 \to \mathcal{C}_2$ as follows. For $C \in \mathcal{C}_1$ set

$$\theta[C](u \otimes v \otimes w) = \mathbf{k}(v) C(u \otimes w). \tag{37}$$

We define the comultiplication $m^* : \mathcal{E}_1 \to \mathcal{E}_2$ as a pullback of the mapping θ with respect to the isomorphisms ψ_1, ψ_2:

$$m^* := \psi_2^{-1} \circ \theta \circ \psi_1.$$

Assume that $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $\mathcal{E}_2$, i.e., $m^* F = \iota^* G$ in $\mathcal{E}_2$. This is equivalent to the condition that $\psi_2(m^* F) = \psi_2(\iota^* G)$ in $\mathcal{C}_2$, where $\iota^* : C^\infty(\Sigma^2, \Lambda^2) \to \mathcal{E}_2$ is the quotient mapping. On the one hand, $\psi_2(\iota^* G) = \chi_2(G)$. On the other hand, $\psi_2(m^* F) = \theta[\psi_1(F)] = \theta[\chi_1(F)]$. Thus F and G agree on $\mathcal{E}_2$ iff

$$\langle\langle G \rangle\rangle(u \otimes v \otimes w) = \mathbf{k}(v) \langle\langle F \rangle\rangle(u \otimes w) \tag{38}$$

for any $u, v, w \in \mathcal{U}(\mathfrak{g})$.

Now we will check formula (5), i.e., Axiom (FA1). For $f \in C^\infty(M)$ we need to show that $m^*(Sf) = \iota^*(Sf \otimes 1)$ or, equivalently, that for

$u, v, w \in \mathcal{U}(\mathfrak{g})$

$$\langle\langle Sf \otimes 1\rangle\rangle(u \otimes v \otimes w) = \mathbf{k}(v)\langle\langle Sf\rangle\rangle(u \otimes w). \tag{39}$$

An easy calculation with the use of formulas (27) and (33) shows that both sides of (39) equal $\mathbf{h}(u)\mathbf{k}(v)\mathbf{k}(w)f$, whence the statement follows. Formula (5) can be checked similarly.

Axiom (FA3), i.e., the identity $ES = \mathrm{id}_{C^\infty(M)}$, is a part of the definition of a formal strict symplectic realization of the Poisson manifold M, and the target mapping T was constructed to satisfy the identity $ET = \mathrm{id}_{C^\infty(M)}$, which is Axiom (FA4).

Our next goal is to check formula (7), i.e., Axiom (FA5). We start with a pair of functions $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ which agree on $\mathcal{E}_2$, i.e., satisfy condition (38).

We need to check that formula (7) holds. Applying the isomorphism χ to the both sides of formula (7), we obtain an equivalent condition:

$$\langle\langle F\rangle\rangle(u \otimes w) = E\big(\lambda(u)\rho(v)(\delta^* \circ (SE \otimes 1))G\big). \tag{40}$$

It is straightforward that

$$(\lambda(u)\rho(v)) \circ \delta^* = \sum_{i,j} \delta^* \circ \Big(\big((\lambda(u_i')\rho(v_j')\big) \otimes \big((\lambda(u_i'')\rho(v_j'')\big)\Big).$$

Then, using the fact that

$$\lambda(u) \circ S = S \circ \mathbf{h}(u), \;\; \rho(u) \circ S = S \circ \mathbf{k}(u),$$

and Lemma (6), we see that

$$(\lambda(u)\rho(v)) \circ (SE) = \epsilon_{\mathcal{U}}(v) \cdot \sum_i (SE) \circ (\lambda(u_i')\rho(u_i'')).$$

Finally, taking into account that $E \circ \delta^* = E_2$, $E_2 \circ (SE \otimes 1) = E_2$, and formula (27), we obtain that

$$\langle\langle F\rangle\rangle(u \otimes v) = \sum_{i,j} \mathbf{k}(v_j')E_2\big((\lambda(u_i')\rho(u_i'')) \otimes (\lambda(u_i''')\rho(v_j''))G\big) =$$

$$\sum_i E_2\big((\lambda(u_i')\rho(u_i'')) \otimes (\lambda(u_i''')\rho(v))G\big) = \sum_i \langle\langle G\rangle\rangle(u_i' \otimes u_i'' \otimes v),$$

where we have used the following notation:

$$\big((\delta_{\mathcal{U}} \otimes 1) \circ \delta_{\mathcal{U}}\big)(u) = \big((1 \otimes \delta_{\mathcal{U}}) \circ \delta_{\mathcal{U}}\big)(u) = \sum_i u_i' \otimes u_i'' \otimes u_i'''.$$

Thus condition (40) is equivalent to the following one:

$$\langle\langle F\rangle\rangle(u \otimes v) = \sum_i \langle\langle G\rangle\rangle(u_i' \otimes u_i'' \otimes v). \tag{41}$$

Formula (41) is an immediate consequence of (38).

The remaining axioms of a formal symplectic groupoid can be checked along the same lines.

In order to check Property P of the comultiplication we need the following lemma.

Lemma 8. *If elements $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $\mathcal{E}_2$, then for any $u, v, w \in \mathcal{U}(\mathfrak{g})$ the elements $\tilde{F} = \epsilon_{\mathcal{U}}(v) \cdot (\lambda(u)\rho(v)F)$ and $\tilde{G} = \lambda_2^0(u)\lambda_2^1(v)\lambda_2^2(w)G$ agree on $\mathcal{E}_2$ as well.*

Proof. We have to show that

$$\langle\langle \tilde{G} \rangle\rangle(\tilde{u} \otimes \tilde{v} \otimes \tilde{w}) = \mathbf{k}(\tilde{v})\langle\langle \tilde{F} \rangle\rangle(\tilde{u} \otimes \tilde{w})$$

for any $\tilde{u}, \tilde{v}, \tilde{w} \in \mathcal{U}(\mathfrak{g})$. It follows immediately from the fact that the representations $\lambda_n^k,\ 0 \leq k \leq n$, pairwise commute. □

Assume that elements $F_i \in C^\infty(\Sigma, \Lambda)$ and $G_i \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $\mathcal{E}_2$ for $i = 1, 2$. To check Property P we need to prove that

$$\langle\langle \{G_1, G_2\}_{\Sigma^2} \rangle\rangle(u \otimes v \otimes w) = \mathbf{k}(v)\langle\langle \{F_1, F_2\}_\Sigma \rangle\rangle(u \otimes w).$$

A straightforward calculation with the use of formulas (13), (29), and (28) applied to condition (38) with $F = F_i, G = G_i$, where $i = 1, 2$, shows that

$$E\left(\{F_1, F_2\}_\Sigma\right) = E_2\left(\{G_1, G_2\}_{\Sigma^2}\right).$$

Then it remains to use the Jacobi identity and Lemma 8.

In order to check the coassociativity of the comultiplication m^* we consider the mappings

$$\left(m_2^1\right)^*, \left(m_2^2\right)^* : \mathcal{E}_2 \to \mathcal{E}_3$$

induced by $m^* \otimes 1$ and $1 \otimes m^*$ as in Section 2. These mappings are well defined due to Axioms (FA1) and (FA2) given by formulas (5) and (6) respectively. Pushing forward the mappings $(m_2^1)^*$ and $(m_2^2)^*$ via the isomorphisms ψ_2, ψ_3 we obtain the mappings $\theta_2^1, \theta_2^2 : \mathcal{C}_2 \to \mathcal{C}_3$ such that

$$\theta_2^1 = \psi_3 \circ \left(m_2^1\right)^* \circ \psi_2^{-1},\ \theta_2^2 = \psi_3 \circ \left(m_2^2\right)^* \circ \psi_2^{-1}.$$

These mappings act on an element $C \in \mathcal{C}_2$ as follows:

$$\theta_2^1[C](u \otimes v \otimes w \otimes z) = \mathbf{k}(v)C(u \otimes w \otimes z),$$
$$\theta_2^2[C](u \otimes v \otimes w \otimes z) = \mathbf{k}(w)C(u \otimes v \otimes z).$$

Now, both $\theta_2^1 \circ \theta$ and $\theta_2^2 \circ \theta$ map $B \in \mathcal{C}_1$ to an element $D \in \mathcal{C}_3$ such that

$$D(u \otimes v \otimes w \otimes z) = \mathbf{k}(v)\mathbf{k}(w)B(u \otimes z),$$

which implies the coassociativity of the coproduct m^*.

Assume that there is given a formal symplectic groupoid on (Σ, Λ) over the Poisson manifold M with the source mapping S and comultiplication m^*. To conclude the proof of Theorem 1 we need to prove the following statements.

Lemma 9. *If elements $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $\mathcal{E}_2$, then $E(F) = E_2(G)$.*

Proof. Axiom (FA5) given by (7) and formula (13) imply that

$$E(F) = E(\delta^* \circ (SE \otimes 1))G) = E_2((SE \otimes 1))G) = E_2(G).$$

□

Proposition 2. *The mapping $\psi_2 \circ m^* \circ \psi_1^{-1}$ coincides with the mapping θ, given by formula (37)*

Proof. Axiom (FA1) of a formal symplectic groupoid given by formula (5) means that the formal functions Sf and $Sf \otimes 1$ agree for all $f \in C^\infty(M)$. Similarly, Axiom (FA2) given by formula (6) means that Tf agrees with $1 \otimes Tf$. Finally, zero constant 0 agrees with $1 \otimes Sf - Tf \otimes 1$, since the function $1 \otimes Sf - Tf \otimes 1$ is in the ideal $\mathcal{I}_2$ which is the kernel of the mapping ι^*. Property P implies that if $F \in C^\infty(\Sigma, \Lambda)$ agrees with $G \in C^\infty(\Sigma^2, \Lambda^2)$, then

$$m^*\big(\lambda(f)F\big) = \iota^*\big(\lambda_2^0(f)G\big),\ m^*\big(\rho(f)F\big) = \iota^*\big(\lambda_2^2(f)G\big), \iota^*\big(\lambda_2^1(f)G\big) = 0.$$

Thus for $u, v, w \in \mathcal{U}(\mathfrak{g})$

$$\epsilon_{\mathcal{U}}(v) m^*\big(\lambda(u)\rho(w)F\big) = \iota^*\big(\lambda_2^0(u)\lambda_2^1(v)\lambda_2^2(w)G\big). \tag{42}$$

Taking into account Lemma 9 we obtain from (42) that

$$\mathbf{k}(v)\langle\langle F\rangle\rangle(u \otimes w) = \langle\langle G\rangle\rangle(u \otimes v \otimes w),$$

whence the Proposition follows. □

Proposition 2 shows that the comultiplication m^* is uniquely defined by the source mapping S. This concludes the proof of Theorem 1.

Remark. Let M be a symplectic manifold. Denote by $\bar{M}$ a copy of the manifold M endowed with the opposite symplectic structure and by M_{diag} the diagonal of $M \times \bar{M}$. It follows from the results obtained in [14] that, given a formal symplectic groupoid $\mathbf{G}$ on (Σ, Λ) over a symplectic manifold M with the source mapping S and target mapping T, then the mapping

$$S \otimes T : C^\infty(M \times \bar{M}, M_{diag}) \to C^\infty(\Sigma, \Lambda)$$

is a formal symplectic isomorphism. It can be easily checked that the mapping $S \otimes T$ induces an isomorphism of the formal pair symplectic groupoid on $(M \times \bar{M}, M_{diag})$ over M with the groupoid $\mathbf{G}$.

4. Isomorphisms of formal symplectic groupoids

Let Σ be a symplectic manifold and Λ its Lagrangian submanifold which is a copy of a given Poisson manifold M. In this section we will consider the formal symplectic groupoids on the formal neighborhood (Σ, Λ) over M. It is known that there exists a local symplectic groupoid over M defined on a symplectic manifold Σ'. Its unit space Λ' is a copy of M. One can find a symplectomorphism of a neighborhood V of Λ in Σ onto a neighborhood V' of Λ' in Σ' which identifies Λ with Λ'. One can then transfer the local symplectic groupoid on V' to V and induce a formal symplectic groupoid on (Σ, Λ) over M. We are going to describe the space of all formal symplectic groupoids on (Σ, Λ) over M as a principal homogeneous space of a certain pronilpotent infinite dimensional Lie group.

Let $\mathbf{G}$ and $\mathbf{G}'$ be two formal symplectic groupoids on (Σ, Λ) over M with the source mappings

$$S, S' : C^\infty(M) \to C^\infty(\Sigma, \Lambda),$$

target mappings T, T', and inverse mappings I, I' respectively. Denote by $\chi, \chi' : C^\infty(\Sigma, \Lambda) \to \mathcal{C}$ and by $\chi_1, \chi_1' : C^\infty(\Sigma, \Lambda) \to \mathcal{C}_1$ the corresponding Poisson isomorphisms, as introduced in Section 3. For $F \in C^\infty(\Sigma, \Lambda)$ we use the notations $\langle F \rangle = \chi(F)$, $\langle F \rangle' = \chi'(F)$. There exists a unique Poisson automorphism Q of $C^\infty(\Sigma, \Lambda)$ such that

$$\chi' = \chi \circ Q.$$

It follows from formulas (24) and (25) that for $f \in C^\infty(M)$

$$\langle Sf \rangle = \langle S'f \rangle' \text{ and } \langle Tf \rangle = \langle T'f \rangle',$$

whence

$$S = Q \circ S' \text{ and } T = Q \circ T'. \tag{43}$$

The isomorphisms $\chi_1, \chi_1' : C^\infty(\Sigma, \Lambda) \to \mathcal{C}_1$ push forward the corresponding inverse mappings I and I' of the formal symplectic groupoids $\mathbf{G}, \mathbf{G}'$ to the same mapping $C \mapsto C^\dagger$ on $\mathcal{C}_1$. Therefore

$$QI' = IQ.$$

We want to descibe the structure of the automorphism Q. The isomorphisms χ, χ' respect the filtrations on $C^\infty(\Sigma, \Lambda)$ and $\mathcal{C}$. Therefore, the automorphism Q respects the filtration on $C^\infty(\Sigma, \Lambda)$, i.e., $Q(\mathcal{J}^n) \subset \mathcal{J}^n, n \geq 0$, where $\mathcal{J} = I_\Lambda / I_\Lambda^\infty$ is the kernel of the unit mapping $E : C^\infty(\Sigma, \Lambda) \to C^\infty(M)$ and $\mathcal{J}^0 := C^\infty(\Sigma, \Lambda)$. One can prove a stronger statement.

Proposition 3. *The operator $Q-1 : C^\infty(\Sigma, \Lambda) \to C^\infty(\Sigma, \Lambda)$ increases the filtration degree by one, i.e., $(Q-1)\mathcal{J}^n \subset \mathcal{J}^{n+1}, n \geq 0$.*

Proof. For an arbitrary element $G \in \mathcal{J}^n$ set $F = Q(G) \in \mathcal{J}^n$. We have that $\langle F \rangle = \langle G \rangle'$ and $\langle F \rangle_k = \langle G \rangle_k = \langle G \rangle'_k = 0$ for all $k < n$. According to Lemma 5, $\langle G \rangle_n = \langle G \rangle'_n$, whence $\langle F \rangle_k = \langle G \rangle_k$ for all $k \leq n$. Therefore $(Q-1)G = F - G \in \mathcal{J}^{n+1}$, which concludes the proof. ☐

For $G \in C^\infty(\Sigma, \Lambda)$ set $F = Q(G)$. Using that $\chi(F) = \chi'(G)$, it is easy to check that in standard local Darboux coordinates (x, ξ) on Σ

$$E(\partial^\alpha F) = \Phi_\gamma^{\alpha\beta}(x)\partial_\beta E(\partial^\gamma G),$$

where α, β, γ are multi-indices (recall that $\partial_i = \partial/\partial x^i$ and $\partial^j = \partial/\partial \xi_j$). We see that locally $Q = \Psi^\alpha_\beta(x, \xi)\partial_\alpha \partial^\beta$, i.e., Q is a formal differential operator on the formal neighborhood (Σ, Λ). Proposition 3 implies that the operator

$$H := \log Q = \log\big(1 + (Q-1)\big) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(Q-1)^n.$$

on $C^\infty(\Sigma, \Lambda)$ is correctly defined via a $\mathcal{J}$-adically convergent series and increases the filtration degree by one. Since Q is a Poisson automorphism of $C^\infty(\Sigma, \Lambda)$, the operator H is a derivation of $C^\infty(\Sigma, \Lambda)$ which respects the Poisson bracket. The operator H is a formal vector field on (Σ, Λ) locally given by the formula

$$H = a^i(x, \xi)\partial_i + b_j(x, \xi)\partial^j, \tag{44}$$

where $a^i = 0 \pmod{\xi}$ and $b_j = 0 \pmod{\xi^2}$, since H increases the filtration degree by one. We want to show that H is a formal Hamiltonian vector field on (Σ, Λ).

Lemma 10. *A formal vector field H on (Σ, Λ) respects the Poisson bracket $\{\cdot, \cdot\}_\Sigma$ and increases by one the filtration degree in $C^\infty(\Sigma, \Lambda)$ if and only if there exists a formal Hamiltonian $F \in \mathcal{J}^2$ such that $H = H_F$. If $H = H_F$ for some formal Hamiltonian $F \in \mathcal{J}^2$, then F is defined uniquely.*

Proof. Assume that H is given in standard Darboux coordinates by formula (44). The condition that H respects the Poisson bracket $\{\cdot, \cdot\}_\Sigma$ can be expressed in local coordinates as follows:

$$\partial^i a^j = \partial^j a^i, \ \partial_i b_j = \partial_j b_i, \ \partial_i a^j = -\partial^j b_i,$$

which is equivalent to the fact that the formal one-form $A = a^i d\xi_i - b_j dx^j$ is closed. Introduce a grading $|\cdot|$ on the differential forms in

the variables x, ξ such that $|x| = 0, |dx| = 0, |\xi| = 1, |d\xi| = 1$. The differential $d = \partial_i dx^i + \partial^j d\xi_j$ respects the grading. Denote by A_q the homogeneous component of degree q of the form A. Then

$$A_q = a^i_{q-1} d\xi_i - b_{jq} dx^j, \tag{45}$$

where a^i_q and b_{jq} denote the homogeneous components of a^i and b_j of degree q, respectively. Since $a^i = 0 \pmod{\xi}$ and $b_j = 0 \pmod{\xi^2}$, we see from formula (45) that the series $A = \sum A_q$ starts with the term A_2. The form A is closed iff each homogeneous component A_q is closed. Using the standard homotopy argument involving the Euler operator $\xi_j \partial^j$ related to the grading, we get that if A_q is closed, there exists a unique function $F_q(x, \xi)$ homogeneous of degree q in ξ such that $A_q = dF_q$. Now, $F = F_2 + F_3 + \dots$ is the unique element of $\mathcal{J}^2$ such that $A = dF$, or, equivalently, such that $H = H_F$. □

It follows from Lemma 10 that there exists a unique formal function $F \in \mathcal{J}^2$ such that $Q = \exp H_F$. Now assume that $\mathbf{G}$ is a formal symplectic groupoid on (Σ, Λ) over M with the source mapping S.

Lemma 11. *Let W be a Poisson automorphism of $C^\infty(\Sigma, \Lambda)$ such that $E \circ W = E$ and $W \circ S = S$. Then W is the identity automorphism, $W = 1$.*

Proof. Since W is a Poisson automorphism of $C^\infty(\Sigma, \Lambda)$ and $W \circ S = S$, we get for $f \in C^\infty(M)$ and $F \in C^\infty(\Sigma, \Lambda)$ that $W(H_{Sf} F) = W(\{Sf, F\}_\Sigma) = \{WSf, WF\}_\Sigma = \{Sf, WF\}_\Sigma = H_{Sf} W(F)$. Therefore $W \circ \lambda(u) = \lambda(u) \circ W$ for any $u \in \mathcal{U}(\mathfrak{g})$. Taking into account that $E \circ W = E$, we obtain that

$$\langle F \rangle(u) = E(\lambda(u)F) = E(W\lambda(u)F) = E(\lambda(u)WF) = \langle W(F) \rangle(u).$$

Proposition 1 implies that $W = 1$, which concludes the proof. □

Take an arbitrary element $F \in \mathcal{J}^2$. The operator H_F on $C^\infty(\Sigma, \Lambda)$ increases the filtration degree by one, therefore there is a Poisson automorphism $Q = \exp H_F$ of $C^\infty(\Sigma, \Lambda)$ such that $E \circ Q = E$. The mapping S' uniquely determined by the equation $S = Q \circ S'$ is a Poisson morphism from $C^\infty(M)$ to $C^\infty(\Sigma, \Lambda)$ with the property that $ES' = \mathrm{id}_{C^\infty(M)}$. Therefore it determines a unique formal symplectic groupoid $\mathbf{G}'$ on (Σ, Λ) over M whose source mapping is S'. Take $F' \in \mathcal{J}^2$ and set $Q' = \exp H_{F'}$. Lemma 11 implies that if $S = Q' \circ S'$, then $Q = Q'$ and $F = F'$.

The automorphism Q such that $S = Q \circ S'$ plays the role of the equivalence morphism of the groupoids $\mathbf{G}$ and $\mathbf{G}'$.

Denote by $\mathfrak{g}_\Sigma$ the pronilpotent Lie algebra $(\mathcal{J}^2, \{\cdot,\cdot\}_\Sigma)$ and by $G_\Sigma = \exp \mathfrak{g}_\Sigma$ the corresponding pronilpotent Lie group. The results of this Section can be combined in the following theorem.

Theorem 3. *The space of formal symplectic groupoids over a Poisson manifold M defined on the formal symplectic neighborhood (Σ, Λ) of a Lagrangian submanifold Λ of a symplectic manifold Σ is a principal homogeneous space of the group G_Σ of formal symplectic automorphisms of $C^\infty(\Sigma, \Lambda)$.*

Let $\mathbf{G}$ be a formal symplectic groupoid over a Poisson manifold M defined on the formal neighborhood (T^*M, Z) of the zero section Z of the cotangent bundle T^*M. Denote by τ the antisymplectic involutive automorphism of T^*M given by the formula $\tau : (x, \xi) \mapsto (x, -\xi)$, where $\{x^i\}$ are local coordinates on M lifted to T^*M and $\{\xi_j\}$ the dual fibre coordinates on T^*M. It induces the dual antisymplectic involutive morphism $\tau^* : C^\infty(T^*M, Z) \to C^\infty(T^*M, Z)$. Let S, T, I be the source, target, and inverse mappings of the groupoid $\mathbf{G}$, respectively. Since $T : C^\infty(M) \to C^\infty(T^*M, Z)$ is an anti-Poisson morphism such that $ET = \mathrm{id}_{C^\infty(M)}$, the mapping

$$\tilde{S} = \tau^* \circ T \tag{46}$$

is a Poisson morphism from $C^\infty(M)$ to $C^\infty(T^*M, Z)$ such that $E\tilde{S} = \mathrm{id}_{C^\infty(M)}$. Therefore there exists a unique formal symplectic groupoid $\tilde{\mathbf{G}}$ on (T^*M, Z) over M whose source mapping is $\tilde{S}$. We call $\tilde{\mathbf{G}}$ *the dual formal symplectic groupoid of* $\mathbf{G}$. Theorem 3 implies that there exists a unique symplectic automorphism $Q \in G_\Sigma$ such that

$$S = Q \circ \tilde{S}. \tag{47}$$

The automorphism Q is uniquely represented as $Q = \exp H_F$ for some element $F \in \mathcal{J}^2$. Since $T = IS$, we get from formulas (46) and (47) that

$$S = Q \circ \tau^* \circ I \circ S.$$

Set $W := Q \circ \tau^* \circ I$. One can check that $E \circ Q = E$, $E \circ I = E$, and $E \circ \tau^* = E$, whence $E \circ W = E$. Since W is a Poisson automorphism of $C^\infty(T^*M, Z)$, it follows from Lemma 11 that $Q \circ \tau^* \circ I = W = 1$. Taking into account that the inverse mapping I is involutive, we obtain that

$$I = Q \circ \tau^* = \exp H_F \circ \tau^*. \tag{48}$$

The Hamiltonian F is canonically related to the formal groupoid $\mathbf{G}$. Since τ^* is involutive, we get that $Q \circ \tau^* = \tau^* \circ Q^{-1}$, whence $H_F \circ \tau^* =$

$-\tau^* \circ H_F$, which means that

$$\tau^* F = F, \tag{49}$$

i.e., that $F(x,\xi) = F(x,-\xi)$.

5. Canonical formal symplectic groupoid of a natural deformation quantization

Let $(M, \{\cdot,\cdot\}_M)$ be a Poisson manifold. Denote by $C^\infty(M)[[\nu]]$ the space of formal series in ν with coefficients from $C^\infty(M)$. As introduced in [1], a formal differentiable deformation quantization on M is an associative algebra structure on $C^\infty(M)[[\nu]]$ with the ν-linear and ν-adically continuous product $*$ (named star-product) given on $f, g \in C^\infty(M)$ by the formula

$$f * g = \sum_{r=0}^{\infty} \nu^r C_r(f,g), \tag{50}$$

where C_r, $r \geq 0$, are bidifferential operators on M, $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = \{f,g\}$. We adopt the convention that the unity of a star-product is the unit constant. Two differentiable star-products $*, *'$ on a Poisson manifold $(M, \{\cdot,\cdot\}_M)$ are called equivalent if there exists an isomorphism of algebras $B : (C^\infty(M)[[\nu]], *') \to (C^\infty(M)[[\nu]], *)$ of the form $B = 1 + \nu B_1 + \nu^2 B_2 + \ldots$, where $B_r, r \geq 1$, are differential operators on M. The existence and classification problem for deformation quantization was first solved in the non-degenerate (symplectic) case (see [5], [20], [8] for existence proofs and [9], [18], [7], [2], [24] for classification) and then Kontsevich [17] showed that every Poisson manifold admits a deformation quantization and that the equivalence classes of deformation quantizations can be parameterized by the formal deformations of the Poisson structure.

All the explicit constructions of star-products enjoy the following property: for all $r \geq 0$ the bidifferential operator C_r in (50) is of order not greater than r in each argument (most important examples are Fedosov's star-products on symplectic manifolds and Kontsevich's star-product on $\mathbb{R}^n$ endowed with an arbitrary Poisson bracket). The star-products with this property were called natural by Gutt and Rawnsley in [10], where general properties of such star-products were studied.

Let $\mathcal{D} = \mathcal{D}(M)$ be the algebra of differential operators with smooth comlex-valued coefficients and $\mathcal{D}[[\nu]]$ be the algebra of formal differential operators on M. The algebra $\mathcal{D}$ has a natural increasing filtration $\{\mathcal{D}_r\}$, where $\mathcal{D}_r$ is the space of differential operators of order not greater than r. We call a formal differential operator $A = A_0 + \nu A_1 + \cdots \in$

$\mathcal{D}[[\nu]]$ *natural* if $A_r \in \mathcal{D}_r$ for any $r \geq 0$. The natural formal differential operators form an algebra which we denote by $\mathcal{N}$.

Let T^*M be the cotangent bundle of the manifold M and Z be its zero section. Denote by $\epsilon : M \to T^*M$ the composition of the identifying mapping from M onto Z with the inclusion mapping of Z into T^*M. It induces the dual mapping $E : C^\infty(T^*M, Z) \to C^\infty(M)$.

If $\{x^k\}$ are local coordinates on M and $\{\xi_k\}$ are the dual fibre coordinates on T^*M, then the principal symbol of an operator $A \in \mathcal{D}_r$, whose leading term is $a^{i_1 \ldots i_r}(x)\partial_{i_1} \ldots \partial_{i_r}$, is given by the formula $\mathrm{Symb}_r(A) = a^{i_1 \ldots i_r}(x)\xi_{i_1} \ldots \xi_{i_r}$. It is globally defined on T^*M and fibrewise is a homogeneous polynomial of degree r. We define a σ-symbol of a natural formal differential operator $A = A_0 + i\nu A_1 + (i\nu)^2 A_2 + \ldots$ as the formal series $\sigma(A) = \mathrm{Symb}_0(A_0) + \mathrm{Symb}_1(A_1) + \ldots$. Such a formal series can be treated as a formal function from $C^\infty(T^*M, Z)$. The mapping $\sigma : A \mapsto \sigma(A)$ is an algebra morphism from $\mathcal{N}$ to $C^\infty(T^*M, Z)$. Moreover, for $A, B \in \mathcal{N}$ the operator $\frac{1}{\nu}[A, B]$ is also natural and

$$\sigma\left(\frac{1}{\nu}[A, B]\right) = \{\sigma(A), \sigma(B)\}_{T^*M}, \tag{51}$$

where $\{\cdot, \cdot\}_{T^*M}$ denotes the standard Poisson bracket on T^*M and the induced bracket on $C^\infty(T^*M, Z)$ given locally by the formula

$$\{\Phi, \Psi\}_{T^*M} = \partial^i\Phi\, \partial_i\Psi - \partial^i\Psi\, \partial_i\Phi.$$

For $f, g \in C^\infty(M)[[\nu]]$ denote by L_f and R_g the operators of $*$-multiplication by f from the left and of $*$-multiplication by g from the right respectively, so that $L_f g = f * g = R_g$. The associativity of $*$ is equivalent to the fact that $[L_f, R_g] = 0$. A star-product $*$ on M is natural iff for any $f, g \in C^\infty(M)[[\nu]]$ the operators L_f, R_g are natural. It was proved in [14] that the mappings

$$S, T : C^\infty(M) \to C^\infty(T^*M, Z)$$

defined by the formulas

$$Sf = \sigma(L_f),\ Tf = \sigma(R_f),$$

where $f \in C^\infty(M)$, are a Poisson and an anti-Poisson morphisms, respectively, which satisfy the formulas $ES = \mathrm{id}_{C^\infty(M)}$ and $ET = \mathrm{id}_{C^\infty(M)}$. Moreover, for $f, g \in C^\infty(M)$ the formal functions Sf, Tg Poisson commute. For each natural deformation quantization on M we constructed in [15] an involutive antisymplectic automorphism I of the Poisson algebra $C^\infty(T^*M, Z)$ such that $IS = T$ and $IT = S$. It follows from Theorem 1 that there exists a canonical formal symplectic groupoid on (T^*M, Z) over M with the source mapping S, target

mapping T, and inverse mapping I. We call it *the formal symplectic groupoid of the natural deformation quantization*.

If $*$ and $*'$ are two equivalent natural star products on M, it was proved in [10] that any equivalence operator B of these star products satisfying the identity

$$Bf * Bg = B(f *' g)$$

can be represented as $B = \exp \frac{1}{\nu} X$, where X is a natural operator such that $X = 0 \pmod{\nu^2}$. Let $\mathbf{G}$ and $\mathbf{G}'$ be the formal symplectic groupoids of the star products $*$ and $*'$ with the source mappings S and S', respectively. It is easy to check that if Q is the equivalence morphism of these groupoids such that $S = Q \circ S'$, then

$$Q = \exp H_{\sigma(X)}.$$

6. Deformation quantizations with separation of variables

Let M be a complex manifold endowed with a Poisson tensor η of type (1,1) with respect to the complex structure. We call such manifolds Kähler-Poisson. If η is nondegenerate, M is a Kähler manifold.

If $U \subset M$ is a coordinate chart with local holomorphic coordinates $\{z^k, \bar{z}^l\}$, we will write $\eta = g^{\bar{l}k} \bar{\partial}_l \wedge \partial_k$ on U, where $\partial_k = \partial/\partial z^k$ and $\bar{\partial}_l = \partial/\partial \bar{z}^l$. The condition that η is a Poisson tensor is expressed in terms of $g^{\bar{l}k}$ as follows:

$$g^{\bar{l}k} \partial_k g^{\bar{n}m} = g^{\bar{n}k} \partial_k g^{\bar{l}m} \text{ and } g^{\bar{l}k} \bar{\partial}_l g^{\bar{n}m} = g^{\bar{l}m} \bar{\partial}_l g^{\bar{n}k}. \tag{52}$$

The corresponding Poisson bracket on M is given locally as

$$\{\phi, \psi\}_M = g^{\bar{l}k} (\bar{\partial}_l \phi \, \partial_k \psi - \bar{\partial}_l \, \psi \partial_k \phi). \tag{53}$$

We say that a star-product (50) on a Kähler-Poisson manifold M defines a deformation quantization with separation of variables on M if the bidifferential operators C_r differentiate their first argument in antiholomorphic directions and its second argument in holomorphic ones.

With the assumption that the unit constant 1 is the unity of the star-algebra $(C^\infty(M)[[\nu]], *)$, the condition that $*$ is a star-product with separation of variables can be restated as follows. For any local holomorphic function a and antiholomorphic function b the operators L_a and R_b are the operators of point-wise multiplication by the functions a and b respectively, $L_a = a, \; R_b = b$. In such a case it is easy to check that $C_1(\phi, \psi) = g^{\bar{l}k} \bar{\partial}_l \phi \, \partial_k \psi$, so that

$$\phi * \psi = \phi \psi + \nu g^{\bar{l}k} \bar{\partial}_l \phi \, \partial_k \psi + \ldots \tag{54}$$

Deformation quantizations with separation of variables on a Kähler manifold M (also known as deformation quantizations of the Wick type, see [3]) are completely described and parameterized by the formal deformations of the Kähler form on M in [11]. If $\left(g^{\bar{l}k}\right)$ is an arbitrary matrix with constant entries, the formula

$$(\phi * \psi)(z, \bar{z}) = \left(\exp \nu g^{\bar{l}k} \frac{\partial}{\partial \bar{v}^l} \frac{\partial}{\partial v_k}\right) \phi(z, \bar{v}) \psi(v, \bar{z})|_{v=z, \bar{v}=\bar{z}}$$

defines a star-product with separation of variables on the Kähler-Poisson manifold $(\mathbb{C}^d, g^{\bar{l}k} \bar{\partial}_l \wedge \partial_k)$. One can give more elaborate examples of deformation quantizations with separation of variables on Kähler-Poisson manifolds. We conjecture that star-products with separation of variables exist on an arbitrary Kähler-Poisson manifold and they can be parameterized by the formal deformations of the Kähler-Poisson tensor η (not the equivalence classes, but the star-products themselves). The nature of this parameterization must be very different from that of the parameterization by the formal deformations of the Kähler form in the Kähler case (see also [13]).

For a given star-product with separation of variables $*$ on M there exists a unique formal differential operator B on M such that

$$B(ab) = b * a \tag{55}$$

for any local holomorphic function a and antiholomorphic function b. The operator B is called the formal Berezin transform (see [12]). One can check that the operator Δ defined locally by the formula $g^{\bar{l}k} \partial_k \bar{\partial}_l$ is coordinate invariant and thus globally defined on M and that

$$B = 1 + \nu \Delta + \dots . \tag{56}$$

In particular, B is invertible. Introduce a dual star product $\tilde{*}$ on M by the formula

$$\phi \tilde{*} \psi = B^{-1}(B\psi * B\phi). \tag{57}$$

We will show that $\tilde{*}$ is a deformation quantization with separation of variables on the complex manifold M endowed with the opposite Poisson tensor $-\eta$. This statement was proved in the Kähler case in [12], but the proof does not work in the Kähler-Poisson case.

It follows from (55) that

$$Ba = a \text{ and } Bb = b. \tag{58}$$

In particular, $B1 = 1$.

Lemma 12. *For any local holomorphic function a and antiholomorphic function b*

$$BaB^{-1} = R_a \text{ and } BbB^{-1} = L_b.$$

Proof. We need to show that $BaB^{-1}f = f * a$ for any formal function f. Since B is invertible, the function f can be representad as $f = Bg$ for some formal function g. Now we need to check that $B(ag) = Bg * a$ for an arbitrary formal function g. It suffices to check it only for g of the form $g = \tilde{a}b$, where $\tilde{a}$ is a local holomorphic function and b a local antiholomorphic function. We have

$$B(a\tilde{a}b) = b * (\tilde{a}a) = b * (\tilde{a} * a) = (b * \tilde{a}) * a = B(\tilde{a}b) * a.$$

The formula $BbB^{-1} = L_b$ can be proved similarly. □

Denote by $\tilde{L}_\phi$ the operator of star-multiplication by a function ϕ from the left and by $\tilde{R}_\psi$ the operator of star-multiplication by a function ψ from the right with respect to the star-product $\tilde{*}$. It follows form (57) that

$$\tilde{L}_\phi = B^{-1} R_{B\phi} B \text{ and } \tilde{R}_\psi = B^{-1} L_{B\psi} B. \tag{59}$$

Proposition 4. *The dual star-product $\tilde{*}$ given by formula (57) is a deformation quantization with separation of variables on the manifold M endowed with the same complex structure but with the opposite Poisson tensor $-\eta$.*

Proof. Lemma 12 and formulas (58) and (59) imply that for any local holomorphic function a

$$\tilde{L}_a = B^{-1} R_{Ba} B = B^{-1} R_a B = B^{-1}(BaB^{-1})B = a.$$

Similarly, $\tilde{R}_b = b$ for any local antiholomorphic function b. Thus $\tilde{*}$ is a star-product with separation of variables. Using formulas (54), (56), and (57) we get that

$$\phi \tilde{*} \psi = \phi\psi - \nu g^{\bar{l}k} \bar{\partial}_l \phi \partial_k \psi + \ldots,$$

which implies that $\tilde{*}$ is a star-product on the Kähler-Poisson manifold $(M, -\eta)$. □

Lemma 12 and formula (58) imply that for any local holomorphic functions $a, \tilde{a}$ and antiholomorphic functions $b, \tilde{b}$

$$[BaB^{-1}, \tilde{a}] = [R_a, L_{\tilde{a}}] = 0 \text{ and } [BbB^{-1}, \tilde{b}] = [L_b, R_{\tilde{b}}] = 0. \tag{60}$$

It follows from formula (56) that $B = \exp\left(\frac{1}{\nu} X\right)$ for some formal differential operator

$$X = \nu^2 X_2 + \nu^3 X_3 + \ldots, \tag{61}$$

where $X_2 = \Delta$. We want to show that the operator X is natural. To this end we need the following technical lemma. If U is a holomorphic chart on M with local coordinates $\{z^k, \bar{z}^l\}$ we denote by $\{\zeta_k, \bar{\zeta}_l\}$ the dual fibre coordinates on T^*U and set $\partial^k = \partial/\partial\zeta_k$ and $\bar{\partial}^l = \partial/\partial\bar{\zeta}_l$.

Lemma 13. *Given an integer $n \geq 2$, let X be a nonzero differential operator on a holomorphic chart U with coordinates $\{z^k, \bar{z}^l\}$, such that the operators $[[X, z^i], z^k]$ and $[[X, \bar{z}^j], \bar{z}^l]$ are of order not greater than $n-2$ for any i, j, k, l. Then the operator X is of order not greater than n.*

Proof. Assume that X is a differential operator of order $N > n$. Its principal symbol $p(\zeta, \bar{\zeta})$ is a nonzero homogeneous polynomial of degree N with respect to the fibre coordinates $\{\zeta_k, \bar{\zeta}_l\}$. The condition that the operator $[[X, z^i], z^k]$ is of order not greater than $n-2$ means that the function $\partial^i\partial^k p$ is a polynomial of order not greater than $n-2$ in the variables $\zeta, \bar{\zeta}$. On the other hand, $\partial^i\partial^k p$ is of order $N-2 > n-2$ which means that $\partial^i\partial^k p = 0$ for any i, k. Similarly, $\bar{\partial}^j\bar{\partial}^l p = 0$ for any j, l. Since $N \geq 3$, at least one of the partial derivatives $\partial^i\partial^k p$ or $\bar{\partial}^j\bar{\partial}^l p$ should be nonzero. Thus the assumption that $N > n$ leads to a contradiction. $\square$

Formula (58) implies that for any n the operator X_n in (61) annihilates holomorphic and antiholomorphic functions. In particular, $X_n 1 = 0$. We get from formula (60) that

$$\left[\exp\left(\frac{1}{\nu}\operatorname{ad} X\right)a, \tilde{a}\right] = [BaB^{-1}, \tilde{a}] = 0 \text{ and}$$

$$\left[\exp\left(\frac{1}{\nu}\operatorname{ad} X\right)b, \tilde{b}\right] = [BbB^{-1}, \tilde{b}] = 0. \tag{62}$$

Expanding the left-hand sides of formulas (62) in the formal series in the parameter ν and equating the coefficient at ν^{n-1} to zero, we get

$$\sum_{k\geq 1}\frac{1}{k!}\sum_{i_1+\ldots+i_k-k=n-1}\left[[X_{i_1}, \ldots, [X_{i_k}, a]\ldots], \tilde{a}\right] = 0 \tag{63}$$

for $n \geq 2$. Since all the indices i_j in (63) satisfy the condition $i_j \geq 2$, we have that $n - 1 = i_1 + \ldots + i_k - k \geq k$. Thus we obtain from (63) that

$$\left[[X_n, a], \tilde{a}\right] = -\sum_{k=2}^{n-1}\frac{1}{k!}\sum_{i_1+\ldots+i_k-k=n-1}\left[[X_{i_1}, \ldots, [X_{i_k}, a]\ldots], \tilde{a}\right]. \tag{64}$$

Similarly,

$$\left[[X_n, b]\,,\tilde{b}\right] = -\sum_{k=2}^{n-1} \frac{1}{k!} \sum_{i_1+\ldots+i_k-k=n-1} \left[[X_{i_1}, \ldots, [X_{i_k}, b]\ldots]\,,\tilde{b}\right]. \tag{65}$$

The right-hand sides of equations (64) and (65) depend only on X_k with $k < n$. We know that $X_2 = \Delta$ is of order (not greater than) two. Assume that we have proved that X_k is of order not greater than k for all $k < n$. We see from (64) and (65) that $[[X_n, a], \tilde{a}]$ and $[[X_n, b], \tilde{b}]$ are of order not greater than $n-2$. It follows from Lemma 13 that X_n is of order not greater than n. The induction shows that X is indeed a natural operator. We have proved the following proposition.

Proposition 5. *The formal Berezin transform B of a deformation quantization with separation of variables on a Kähler-Poisson manifold is of the form $B = \exp \frac{1}{\nu} X$, where X is a natural differential operator such that $X = 0 \pmod{\nu^2}$.*

It follows from Proposition 5 that the conjugation of the formal differential operators with respect to the formal Berezin transform, $A \mapsto BAB^{-1}$, leaves invariant the algebra $\mathcal{N}$ of natural differential operators. In particular, the operators $R_a = BaB^{-1}$ and $L_b = BbB^{-1}$ are natural. Now, if $f = ab = a * b$ we see that $L_f = L_{a*b} = L_a L_b = aL_b$ and $R_f = R_{a*b} = R_b R_a = bR_a$ are natural differential operators. Using the same arguments as in Proposition 1 of [15] we can prove the following theorem.

Theorem 4. *Any deformation quantization with separation of variables on a Kähler-Poisson manifold is natural.*

Theorem 4 was proved in [3] and [19] in the Kähler case.

It follows from Theorem 4 that to any deformation quantization with separation of variables on a Kähler-Poisson manifold M there corresponds a canonical formal symplectic groupoid on (T^*M, Z) over M. Since for any deformation quantization with separation of variables $L_a = a$ and $R_b = b$, we see that $Sa = \sigma(L_a) = \sigma(a) = a$ and, similarly, $Tb = b$ (abusing notations we denote by a and b both local functions on M and their lifts to T^*M with respect to the standard bundle projection).

Given a Kähler-Poisson manifold M, we call a formal symplectic groupoid on (T^*M, Z) over M such that $Sa = a$ and $Tb = b$ for any local holomorphic function a and antiholomorphic function b, a *formal symplectic groupoid with separation of variables.*

7. Formal symplectic groupoid with separation of variables

In this section we will show that for any Kähler-Poisson manifold M there is a unique formal symplectic groupoid with sepration of variables over M. Let $U \subset M$ be an arbitrary coordinate chart with local holomorphic coordinates $\{z^k, \bar{z}^l\}$. Introduce differential operators $D^k, \bar{D}^l$ on U by the formulas

$$D^k\psi = g^{\bar{l}k}\bar{\partial}_l\psi = -\{z^k, \psi\}_M \text{ and } \bar{D}^l\psi = g^{\bar{l}k}\partial_k\psi = \{\bar{z}^l, \psi\}_M,$$

where the Poisson bracket $\{\cdot,\cdot\}_M$ is given by formula (53). Conditions (52) are equivalent to the statement that

$$[D^k, D^m] = 0 \text{ and } [\bar{D}^l, \bar{D}^n] = 0 \tag{66}$$

for any k, l, m, n. Using the operators $D^k, \bar{D}^l$ we can write

$$\{\phi, \psi\}_M = D^k\phi\, \partial_k\psi - D^k\psi\, \partial_k\phi = \bar{\partial}_l\phi\, \bar{D}^l\psi - \bar{\partial}_l\psi\, \bar{D}^l\phi. \tag{67}$$

Denote by $\{\zeta_k, \bar{\zeta}_l\}$ the fibre coordinates on T^*U dual to $\{z^k, \bar{z}^l\}$. The standard Poisson bracket on T^*M can be written locally as

$$\{\Phi, \Psi\}_{T^*M} = \partial^k\Phi\, \partial_k\Psi - \partial^k\Psi\, \partial_k\Phi + \bar{\partial}^l\Phi\, \bar{\partial}_l\Psi - \bar{\partial}^l\Psi\, \bar{\partial}_l\Phi, \tag{68}$$

where $\partial^k = \partial/\partial\zeta_k,\ \bar{\partial}^l = \partial/\partial\bar{\zeta}_l$. The Poisson bracket on T^*M induces a Poisson bracket on $C^\infty(T^*M, Z)$ which will be denoted also by $\{\cdot,\cdot\}_{T^*M}$. Introduce mappings $S, T : C^\infty(U) \to C^\infty(T^*U, Z \cap U)$ by the formulas

$$(S\phi)(z, \bar{z}, \zeta) = e^{\zeta_k D^k}\phi,\ (T\psi)(z, \bar{z}, \bar{\zeta}) = e^{\bar{\zeta}_l \bar{D}^l}\psi, \tag{69}$$

where $\phi, \psi \in C^\infty(M)$, the variables $\zeta, \bar{\zeta}$ are used as formal parameters, and the exponentials are defined via formal Taylor series.

Proposition 6. *The mappings*

$$S, T : (C^\infty(U), \{\cdot,\cdot\}_M) \to (C^\infty(T^*U, Z \cap U), \{\cdot,\cdot\}_{T^*M})$$

are a Poisson and an anti-Poisson morphisms, respectively. For any $\phi, \psi \in C^\infty(U)$ the elements $S\phi, T\psi \in C^\infty(T^*U, Z \cap U)$ Poisson commute.

Proof. Since $D^k, \bar{D}^l$ are derivations of the algebra $C^\infty(T^*U, Z \cap U)$, the operators $e^{\zeta_k D^k}, e^{\bar{\zeta}_l \bar{D}^l}$ are automorphisms of this algebra which implies that S, T are algebra homomorphisms. We see from (66) and (69) that

$$\partial^k(S\phi) = D^k(S\phi) \text{ and } \bar{\partial}^l(T\psi) = \bar{D}^l(T\psi). \tag{70}$$

Fix arbitrary functions $\phi, \psi \in C^\infty(U)$ and introduce an element $u(\zeta) \in C^\infty(T^*U, Z \cap U)$ by the formula

$$u(\zeta) = \{S\phi, S\psi\}_{T^*M}.$$

In order to show that S is a Poisson morphism we need to prove that $u(\zeta) = S\{\phi,\psi\}_M = e^{\zeta_k D^k}\{\phi,\psi\}_M$. This amounts to checking that $u(0) = \{\phi,\psi\}_M$ and that $\partial^m u = D^m u$. Using (68), (69), and (70) we get

$$u(\zeta) = \{S\phi, S\psi\}_{T^*M} = \partial^k(S\phi)\partial_k(S\psi) - \partial^k(S\psi)\partial_k(S\phi) \\ = D^k(S\phi)\partial_k(S\psi) - D^k(S\psi)\partial_k(S\phi). \tag{71}$$

It follows from (67) and (71) that

$$u(0) = D^k\phi\,\partial_k\psi - D^k\psi\,\partial_k\phi = \{\phi,\psi\}_M. \tag{72}$$

Now, taking into account (52) and (66), we obtain from (71) that

$$\begin{aligned}\partial^m u - D^m u = \big(&D^m D^k(S\phi)\partial_k(S\psi) - D^m D^k(S\psi)\partial_k(S\phi)+\\ &D^k(S\phi)\partial_k(D^m S\psi) - D^k(S\psi)\partial_k(D^m S\phi)\big)-\\ &\big(D^m D^k(S\phi)\partial_k(S\psi) - D^m D^k(S\psi)\partial_k(S\phi)+\\ &D^k(S\phi)D^m\partial_k(S\psi) - D^k(S\psi)D^m\partial_k(S\phi)\big) =\\ &D^k(S\phi)[\partial_k, D^m](S\psi) - D^k(S\psi)[\partial_k, D^m](S\phi) =\\ &g^{\bar{l}k}\partial_k g^{\bar{n}m}\bar\partial_l(S\phi)\bar\partial_n(S\psi) - g^{\bar{l}k}\partial_k g^{\bar{n}m}\bar\partial_l(S\psi)\bar\partial_n(S\phi) =\\ &g^{\bar{l}k}\partial_k g^{\bar{n}m}\bar\partial_l(S\phi)\bar\partial_n(S\psi) - g^{\bar{n}k}\partial_k g^{\bar{l}m}\bar\partial_l(S\phi)\bar\partial_n(S\psi) = 0,\end{aligned}$$

which concludes the check that S is a Poisson morphism. The proof that T is an anti-Poisson morphism is similar. It remains to show that $\{S\phi, T\psi\}_{T^*M} = 0$. It follows from (68), (69), and (70) that

$$\begin{aligned}\{S\phi, T\psi\}_{T^*M} &= \partial^k S\phi\,\partial_k T\psi - \bar\partial^l T\psi\,\bar\partial_l S\phi = D^k S\phi\,\partial_k T\psi -\\ &\bar D^l T\psi\,\bar\partial_l S\phi = g^{\bar{l}k}\bar\partial_l S\phi\,\partial_k T\psi - g^{\bar{l}k}\partial_k T\psi\,\bar\partial_l S\phi\partial_k = 0.\end{aligned}$$

□

According to Theorem 1 there exists a canonical formal symplectic groupoid $\mathbf{G}_U$ on the formal neighborhood $(T^*U, Z\cap U)$ such that the mappings S, T are the source and target maps for $\mathbf{G}_U$ respectively. The mapping $\tau : (z,\bar z,\zeta,\bar\zeta) \mapsto (z,\bar z,-\zeta,-\bar\zeta)$ is a global anti-Poisson involutive automorphism of T^*M. It induces an anti-Poisson involutive automorphism of the Poisson algebra $C^\infty(T^*M, Z)$. Set $\tilde S = \tau^* T$ and $\tilde T = \tau^* S$. Thus for $\phi,\psi \in C^\infty(U)$

$$(\tilde S\phi)(z,\bar z,\bar\zeta) = e^{-\bar\zeta_l \bar D^l}\phi,\ (\tilde T\psi)(z,\bar z,\zeta) = e^{-\zeta_k D^k}\psi. \tag{73}$$

It follows from Proposition 6 that the mappings

$$\tilde S, \tilde T : (C^\infty(U), \{\cdot,\cdot\}_M) \to (C^\infty(T^*U, Z\cap U), \{\cdot,\cdot\}_{T^*M})$$

are a Poisson and an anti-Poisson morphisms, respectively. Moreover, for any $\phi, \psi \in C^\infty(U)$ the elements $\tilde{S}\phi, \tilde{T}\psi \in C^\infty(T^*U, Z \cap U)$ Poisson commute. Now, there is a canonical formal symplectic groupoid $\tilde{\mathbf{G}}_U$ on $(T^*U, Z \cap U)$ (the dual of $\mathbf{G}_U$) such that the mappings $\tilde{S}, \tilde{T}$ are the source and target maps of $\tilde{\mathbf{G}}_U$, respectively. According to formula (43) there is a unique formal symplectic automorphism Q of $C^\infty(T^*U, Z \cap U)$ such that

$$S = Q\tilde{S} \text{ and } T = Q\tilde{T}. \tag{74}$$

Let $a, \tilde{a}$ be arbitrary holomorphic functions and $b, \tilde{b}$ arbitrary antiholomorphic functions on U. It follows from formulas (69) and (73) that

$$Sa = a,\ Tb = b, \tilde{S}b = b, \text{ and } \tilde{T}a = a, \tag{75}$$

whence we see that $\mathbf{G}_U$ is a formal symplectic groupoid with separation of variables over M and that the dual formal groupoid $\tilde{\mathbf{G}}_U$ is a formal symplectic groupoid with separation of variables with respect to the opposite complex structure on M. Proposition 6, formulas (74) and (75) imply that

$$\begin{gathered} \{Qa, \tilde{a}\}_{T^*M} = \{Q\tilde{T}a, \tilde{a}\}_{T^*M} = \{Ta, S\tilde{a}\}_{T^*M} = 0 \text{ and} \\ \{Qb, \tilde{b}\}_{T^*M} = \{Q\tilde{S}b, \tilde{b}\}_{T^*M} = \{Sb, T\tilde{b}\}_{T^*M} = 0. \end{gathered} \tag{76}$$

We would like to draw the reader's attention to the analogy between formulas (62) and (76). There exists a unique element $F \in \mathcal{J}^2$ such that $Q = \exp H_F$. Represent it as

$$F = F_2 + F_3 + \ldots, \tag{77}$$

where F_q is the homogeneous component of F of degree q with respect to the variables $\zeta_k, \bar{\zeta}_l$. Extracting the homogeneous components of degree $n-2$ of the left-hand sides of (76) and equating them to zero we obtain the following formulas where we drop the subscript T^*M in all the Poisson brackets:

$$\begin{gathered} \{\{F_n, a\}, \tilde{a}\} = -\sum_{k=2}^{n-1} \frac{1}{k!} \sum_{i_1+\ldots+i_k-k=n-1} \{\{F_{i_1}, \ldots, \{F_{i_k}, a\} \ldots\}, \tilde{a}\}, \\ \left\{\{F_n, b\}, \tilde{b}\right\} = -\sum_{k=2}^{n-1} \frac{1}{k!} \sum_{i_1+\ldots+i_k-k=n-1} \left\{\{F_{i_1}, \ldots, \{F_{i_k}, b\} \ldots\}, \tilde{b}\right\}. \end{gathered} \tag{78}$$

The right-hand sides of (78) depend only on F_q for $q < n$ and are assumed to be equal to zero for $n = 2$.

Lemma 14. *Let $\Phi_q = \Phi(z, \bar{z}, \zeta, \bar{\zeta})$ be a homogeneous function of degree q in the variables $\zeta, \bar{\zeta}$ on T^*U such that $\{\{\Phi_q, z^i\}_{T^*M}, z^k\}_{T^*M} = 0$ and $\{\{\Phi_q, \bar{z}^j\}_{T^*M}, \bar{z}^l\}_{T^*M} = 0$ for any i, j, k, l. Then $\Phi_2 = \phi^{\bar{l}k}(z, \bar{z})\zeta_k \bar{\zeta}_l$ for some function $\phi^{\bar{l}k}$ on U and $\Phi_q = 0$ for $q \geq 3$.*

Proof. Using formula (68) we get that $\{\{\Phi_q, z^i\}_{T^*M}, z^k\}_{T^*M} = \partial^i \partial^k \Phi_q = 0$ and $\{\{\Phi_q, \bar{z}^j\}_{T^*M}, \bar{z}^l\}_{T^*M} = \bar{\partial}^j \bar{\partial}^l \Phi_q = 0$, whence the Lemma follows. □

Lemma 14 applied to formulas (78) implies that function (77) is uniquely determined by the term F_2 which is of the form $F_2 = \phi^{\bar{l}k}(z, \bar{z})\zeta_k \bar{\zeta}_l$. We can find F_2 explicitly using formulas (69), (73), and (74). For an arbitrary $f = f(z, \bar{z}) \in C^\infty(U)$ calculate the both sides of the formula $Sf = Q(\tilde{S}f)$ modulo $\mathcal{J}^2$:

$$(1 + \zeta_k D^k) f = (1 + H_{F_2})(1 - \bar{\zeta}_l \bar{D}^l) f \pmod{\mathcal{J}^2}. \tag{79}$$

It follows from formulas (68) and (79) that $\partial^k F_2 = g^{\bar{l}k} \bar{\zeta}_l$, whence we obtain that $\phi^{\bar{l}k} = g^{\bar{l}k}$ and therefore

$$F_2 = g^{\bar{l}k} \zeta_k \bar{\zeta}_l.$$

The remaining terms of series (77) can be found recursively from (78) in local coordinates. Formula (49) implies that $F_k = 0$ for the odd values of k. We conclude that the function F and the automorphism $Q = \exp H_F$ are uniquely determined by the Kähler-Poisson tensor $g^{\bar{l}k}$. Since condition (76) on Q is coordinate independent, both F and Q are globally defined on (T^*M, Z). It follows from formulas (74) and (75) that for $f(z, \bar{z}) = a(z)b(\bar{z})$

$$Sf = S(ab) = Sa \cdot Sb = a \cdot Qb$$

is completely determined by Q which means that the source mapping S is uniquely defined and global on M. The following theorem is a consequence of Theorem 1.

Theorem 5. *For any Kähler-Poisson manifold M there exists a unique formal symplectic groupoid with separation of variables on (T^*M, Z) over M. Its source and target mappings are given locally by formulas (69).*

Now let $*$ be a star product with separation of variables on a Kähler-Poisson manifold M. Theorem 4 states that it is natural. The formal symplectic groupoid of the star product $*$ is the unique formal symplectic groupoid with separation of variables on (T^*M, Z) over M. According to Proposition 5 the formal Berezin transform B of the star product $*$ is of the form $B = \exp \frac{1}{\nu} X$, where X is a natural formal

differential operator on M. Using formula (51) we can derive from (62) and (76) that

$$\sigma(X) = F,$$

where $F = F_2 + F_4 + \dots$ is determined by the condition that $F_2 = \mathrm{Symb}_2(\Delta) = g^{\bar{l}k}\zeta_k\bar{\zeta}_l$ and equations (78).

8. Appendix

In this section we give a proof of Theorem 2. To this end we need some preparations.

Let $K = (i_1, \dots, i_n)$ be a multi-index. Denote by $K' = (i_2, i_1, \dots, i_n)$ the multi-index obtained from K by permuting i_1 and i_2, and by $\tilde{K} = (j_1, \dots, j_n)$ the multi-index such that $j_1 = i_1$ and $j_2 \le \dots \le j_n$ is the ordering permutation of $i_2, \dots, i_n$. If $u^K = u^{i_1 \dots i_n}$ is a tensor symmetric in $i_2, \dots, i_n$ then the tensor

$$v^K = u^K - u^{K'} \tag{80}$$

is skew symmetric in i_1, i_2, symmetric in $i_3, \dots, i_n$ and its cyclic sum over i_1, i_2, i_3 is zero.

Lemma 15. *Suppose that $v^K = v^{i_1 \dots i_n}$ is a tensor skew symmetric in i_1, i_2, symmetric in $i_3, \dots, i_n$ and its cyclic sum over i_1, i_2, i_3 is zero. There exists a unique tensor u^K symmetric in $i_2, \dots, i_n$ that satisfies (80) and such that $u^K = 0$ if $i_1 \le \dots \le i_n$.*

Proof. To define u^K, consider $\tilde{K} = (j_1, \dots, j_n)$. Set $u^K = 0$ if $j_1 \le j_2$ and $u^K = v^{\tilde{K}}$ if $j_1 \ge j_2$ (these conditions agree if $j_1 = j_2$). Thus $u^K = u^{\tilde{K}}$ which implies that u^K is symmetric in $i_2, \dots, i_n$. In order to show that u^K is well defined we need to check condition (80). For the multi-index K in (80) we can assume without loss of generality that $i_2 \le i_1$ and that $i_3 = \min\{i_3, \dots, i_n\}$. If $i_2 \le i_3$ then $u^K = v^K$ and $u^{K'} = 0$, so (80) holds. If $i_3 < i_2$ then $u^K = v^{i_1 i_3 i_2 \dots}$, $u^{K'} = v^{i_2 i_3 i_1 \dots}$ where the order of the remaining indices does not matter. Now (80) holds since the cyclic sum of the tensor v^K over i_1, i_2, i_3 is zero. $\square$

For a coherent family $\{C_n\}$ and any $f_i, \phi \in C^\infty(M)$ one can prove the following formula using Property B.

$$\begin{aligned} C_n(\phi, f_2, \dots, f_n) = C_n(f_2, \dots, f_n, \phi) + \\ \sum_{i=2}^{n} C_{n-1}(f_2, \dots, \{\phi, f_i\}, \dots f_n). \end{aligned} \tag{81}$$

Let $(U, \{x^i\})$ be an arbitrary coordinate chart on M. We will construct an operator C_n locally on U using induction on n. Assume that

one can extend by one element any k-element coherent family for all $k < n$. Consider an n-element coherent family $\{C_k\}, 0 \leq k \leq n-1$. Then for each index i and $k < n-1$ the operators

$$D^i_k(f_1, \ldots, f_k) = C_{k+1}(f_1, \ldots, f_k, x^i)$$

form a coherent family. By induction this family can be extended by an operator D^i_{n-1} so that

$$\begin{aligned} D^i_{n-1}(f_2, \ldots, f_k, f_{k+1}, \ldots f_n) - D^i_{n-1}(f_2, \ldots, f_{k+1}, f_k, \ldots f_n) = \\ C_{n-1}(f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n, x^i), \end{aligned} \tag{82}$$

Introduce the following auxilliary operator

$$\begin{aligned} D_n(f_1, \ldots, f_n) = \Big(D^i_{n-1}(f_2, \ldots, f_n) + \\ \sum_{j=2}^{n} C_{n-1}(f_2, \ldots, \{x^i, f_j\}, \ldots f_n)\Big)\frac{\partial f_1}{\partial x^i}. \end{aligned} \tag{83}$$

The operator D_n annihilates constants and is of order one in the first argument. We will show that for any $k \geq 2$

$$\begin{aligned} D_n(f_1, \ldots, f_k, f_{k+1}, \ldots f_n) - D_n(f_1, \ldots, f_{k+1}, f_k, \ldots f_n) = \\ C_{n-1}(f_1, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n). \end{aligned} \tag{84}$$

Using that a derivation $A(f)$ on U can be written as $A(x^i)\frac{\partial f}{\partial x^i}$, Property A, and formula (82) we can show that equation (84) is a consequence of the following one:

$$\begin{aligned} C_{n-1}(f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n, x^i) + \\ \sum_{j=2}^{k-1} C_{n-2}(f_2, \ldots, \{x^i, f_j\}, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n) + \\ \Big(C_{n-2}(f_2, \ldots, \{\{x^i, f_k\}, f_{k+1}\}, \ldots, f_n) + \\ C_{n-2}(f_2, \ldots, \{f_k, \{x^i, f_{k+1}\}\}, \ldots, f_n)\Big) + \\ \sum_{j=k+2}^{n} C_{n-2}(f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, \{x^i, f_j\}, \ldots, f_n) = \\ C_{n-1}(x^i, f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n). \end{aligned} \tag{85}$$

Using the Jacobi identity, replace the sum in the parentheses in (85) with $C_{n-2}(f_2, \ldots, \{x^i, \{f_k, f_{k+1}\}\}, \ldots, f_n)$. The resulting identity follows from formula (81).

We will construct the operator C_n on the coordinate chart U in the form $C_n = D_n + E_n$, where $E_n(f_1, \ldots, f_n)$ is a multiderivation

symmetric in $f_2, \ldots, f_n$. The operator E_n must be chosen so that C_n would satisfy Property B for $k = 1$ (all other conditions on C_n are already satisfied). This condition can be written in the form

$$V_n(f_1, f_2, \ldots, f_n) = E_n(f_2, f_1, \ldots, f_n) - E_n(f_1, f_2, \ldots, f_n), \tag{86}$$

where the operator V_n is given by the formula

$$\begin{aligned} V_n(f_1, f_2, \ldots, f_n) = D_n(f_1, f_2, \ldots, f_n) - D_n(f_2, f_1, \ldots, f_n) \\ -C_{n-1}(\{f_1, f_2\}, \ldots, f_n). \end{aligned} \tag{87}$$

According to Lemma 15, an operator E_n with the required properties exists if $V_n(f_1, f_2, \ldots, f_n)$ is a multiderivation skew symmetric in f_1, f_2, symmetric in $f_3, \ldots, f_n$, and such that the cyclic sum of V_n over f_1, f_2, f_3 is zero. We will show that the operator V_n enjoys all these properties. Check that the operator V_n is a derivation in the second argument. Substituting formula (83) in (87) and taking into account Property A we see that it remains to check that the operator

$$C_{n-1}(\{x^i, f_2\}, f_3, \ldots, f_n)\frac{\partial f_1}{\partial x^i} - C_{n-1}(\{f_1, f_2\}, \ldots, f_n) \tag{88}$$

is a derivation in f_2. Formula (88) can be rewritten as follows,

$$C_{n-1}(x^j, f_3, \ldots, f_n)\left(\frac{\partial f_1}{\partial x^i}\frac{\partial}{\partial x^j}\{x^i, f_2\} - \frac{\partial}{\partial x^j}\{f_1, f_2\}\right). \tag{89}$$

In local coordinates $\{f, g\} = \eta^{kl}\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial x^l}$, where η^{kl} is a Poisson tensor. The second factor in (89) equals

$$\frac{\partial f_1}{\partial x^k}\frac{\partial}{\partial x^j}\left(\eta^{kl}\frac{\partial f_2}{\partial x^l}\right) - \frac{\partial}{\partial x^j}\left(\eta^{kl}\frac{\partial f_1}{\partial x^k}\frac{\partial f_2}{\partial x^l}\right) = -\eta^{kl}\frac{\partial^2 f_1}{\partial x^j \partial x^k}\frac{\partial f_2}{\partial x^l}.$$

Thus $V_n(f_1, f_2, \ldots, f_n)$ is a derivation in f_2. Since it is obviously skew symmetric in f_1, f_2, it is also a derivation in f_1.

We will prove that $V_n(f_1, f_2, \ldots, f_n)$ is symmetric in $f_3, \ldots, f_n$ using formula (84). For $k \geq 3$

$$\begin{aligned} V_n(f_1, \ldots, f_k, f_{k+1}, \ldots, f_n) - V_n(f_1, \ldots, f_{k+1}, f_k, \ldots, f_n) = \\ C_{n-1}(f_1, f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n) - \\ C_{n-1}(f_2, f_1, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n) - \\ C_{n-2}(\{f_1, f_2\}, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n) = 0. \end{aligned}$$

It remains to show that $V_n(f_1, f_2, \ldots, f_n)$ is a derivation in f_3 and that its cyclic sum over f_1, f_2, f_3 is zero. We have, using formula (84), that

$$\begin{aligned}V_n(f_1, f_2, f_3, \ldots, f_n) = D_n(f_1, f_2, f_3, \ldots, f_n)-\\ D_n(f_2, f_1, f_3, \ldots, f_n) - C_{n-1}(\{f_1, f_2\}, f_3, \ldots, f_n) =\\ D_n(f_1, f_3, f_2, \ldots, f_n) + C_{n-1}(f_1, \{f_2, f_3\}, \ldots, f_n)-\\ D_n(f_2, f_1, f_3, \ldots, f_n) - C_{n-1}(\{f_1, f_2\}, f_3, \ldots, f_n) =\\ D_n(f_1, f_3, f_2, \ldots, f_n) - D_n(f_2, f_1, f_3, \ldots, f_n)+\\ C_{n-2}(\{f_1, \{f_2, f_3\}\}, \ldots, f_n).\end{aligned}$$

We see that the cyclic sum of V_n over f_1, f_2, f_3 is zero due to the Jacobi identity. Therefore,

$$\begin{aligned}V_n(f_1, f_2, f_3, \ldots, f_n) = -V_n(f_2, f_3, f_1, \ldots, f_n)-\\ V_n(f_3, f_1, f_2, \ldots, f_n).\end{aligned} \tag{90}$$

We have already proved that V_n is a derivation in the first two arguments, whence the right hand side and therefore the left hand side of (90) are derivations in f_3. Since $V_n(f_1, \ldots, f_n)$ was shown to be symmetric in $f_3, \ldots, f_n$, this implies that V_n is a multiderivation. This concludes the proof of all the properties of the operator V_n and provides a local construction of the operator C_n. Finally, we use partition of unity to construct C_n globally on M.

References

[1] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D.: Deformation theory and quantization. I. Deformations of symplectic structures. *Ann. Physics* **111** (1978), no. 1, 61 – 110.

[2] Bertelson, M., Cahen, M., and Gutt, S.: Equivalence of star products. Geometry and physics. *Classical Quantum Gravity* **14** (1997), no. 1A, A93 – A107.

[3] Bordemann, M. and Waldmann, S.: A Fedosov star product of the Wick type for Kähler manifolds. *Lett. Math. Phys.* **41** (3) (1997), 243 – 253.

[4] Cattaneo, A.S., Dherin B., and Felder, G.: Formal symplectic groupoid, math.SG/0312380.

[5] De Wilde, M., Lecomte, P.B.A.: Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds. *Lett. Math. Phys.* **7** (1983), no. 6, 487–496.

[6] Coste, A., Dazord, P., and Weinstein, A.: Groupoïdes symplectiques, *Publ. Dép. Math. Nouvelle Sér. A***2** (1987) 1 – 62.

[7] Deligne, P: Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte. *Selecta Math. (N.S.)* **1** (1995), no. 4, 667 – 697.

[8] Fedosov, B.: A simple geometrical construction of deformation quantization. *J. Differential Geom.* **40** (1994), no. 2, 213–238.

[9] Fedosov, B.: *Deformation quantization and index theory.* Mathematical Topics, 9. Akademie Verlag, Berlin, 1996. 325 pp.

[10] Gutt, S. and Rawnsley, J.:Natural star products on symplectic manifolds and quantum moment maps, *Lett. Math. Phys.* **66**(2003), 123 –139.

[11] Karabegov, A.: Deformation quantizations with separation of variables on a Kähler manifold. *Comm. Math. Phys.* **180** (1996), no. 3, 745–755.

[12] Karabegov, A: On the canonical normalization of a trace density of deformation quantization, *Lett. Math. Phys.* **45** (1998), 217 – 228.

[13] Karabegov, A.: A Covariant Poisson Deformation Quantization with Separation of Variables up to the Third Order, *Lett. Math. Phys.* **61** (2002), 255 – 261.

[14] Karabegov, A.: On the dequantization of Fedosov's deformation quantization, *Lett. Math. Phys.* **65** (2003), 133 – 146.

[15] Karabegov, A.: On the inverse mapping of the formal symplectic groupoid of a deformation quantization, math.QA/0403488, to appear in *Lett. Math. Phys.*

[16] Karasev M.V.: Analogues of the objects of Lie group theory for nonlinear Poisson brackets, *Math. USSR Izvestiya* **28** (1987), 497-527.

[17] Kontsevich, M.: Deformation quantization of Poisson manifolds, I, *Lett. Math. Phys.* **66** (2003), 157 – 216 (q-alg/9709040).

[18] Nest, R., Tsygan, B.: Algebraic index theorem. *Comm. Math. Phys.* **172** (1995), no. 2, 223–262.

[19] Neumaier, N.: Universality of Fedosov's Construction for Star Products of Wick Type on Pseudo-Kähler Manifolds. *Rep. Math. Phys.* **52** (2003), 43-80.

[20] Omori, H., Maeda, Y., and Yoshioka, A.: Weyl manifolds and deformation quantization. *Adv. Math.* **85** (1991), no. 2, 224–255.

[21] Vainerman, L.: A note on quantum groupoids. C. R. Acad. Sci. Paris **315** Série I , (1992), 1125 – 1130.

[22] Weinstein, A.: Tangential deformation quantization and polarized symplectic groupoids. Deformation theory and symplectic geometry (Ascona, 1996), 301 – 314, Math. Phys. Stud., **20**, Kluwer Acad. Publ., Dordrecht, 1997.

[23] Weinstein, A.: Symplectic groupoids and Poisson manifolds, *Bull. Amer. Math. Soc.(N.S.)* **16** (1987), 101–103.

[24] Xu, P.: Fedosov $*$-products and quantum momentum maps. *Comm. Math. Phys.* **197** (1998), no. 1, 167–197.

[25] Zakrzewski, S.: Quantum and classical pseudogroups, I and II, *Comm. Math. Phys.* **134** (1990), 347 –395.

(Alexander V. Karabegov) Department of Mathematics and Computer Science, Abilene Christian University, ACU Box 28012, 252 Foster Science Building, Abilene, TX 79699-8012

E-mail address: alexander.karabegov@math.acu.edu